Transfer Function Analysis of Constrained, Distributed Piezoelectric Vibration Energy Harvesting Beam Systems

This paper presents a novel formulation and exact solution of the frequency response function (FRF) of vibration energy harvesting beam systems by the distributed transfer function method (TFM). The method is applicable for coupled electromechanical systems with nonproportional damping, intermediate constraints, and nonclassical boundary conditions, for which the system transfer functions are either very difficult or cumbersome to obtain using available methods. Such systems may offer new opportunities for optimized designs of energy harvesters via parameter tuning. The proposed formulation is also systematic and amenable to algorithmic numerical coding, allowing the system response and its derivatives to be computed by only simple modifications of the parameters in the system operators for different boundary conditions and the incorporation of feedback control principles. Examples of piezoelectric energy harvesters with nonclassical boundary conditions and intermediate constraints are presented to demonstrate the efficacy of the proposed method and its use as a design tool for vibration energy harvesters via tuning of system parameters. The results can also be used to provide benchmarks for assessing the accuracies of approximate techniques. [DOI: 10.1115/1.4038949]

Keywords: distributed transfer function analysis, vibration energy harvesting, piezoelectricity, electromechanical system modeling, constrained systems, energy harvester design

1 Introduction

Over the past two decades, many innovations, particularly in the area of micro-electromechanical systems (MEMS) research, have made it possible to design micro sensors and actuators which are capable of operating with very low power requirements [1], thus allowing sensor networks to be deployed and operated in an autonomous manner [2,3]. Some of these sensors are used in applications with self-contained power sources packaged with the sensors [4–7]. While batteries are currently still used in many of these applications, piezoelectric vibration energy harvesters have shown a lot of potential for use with self-powered devices in applications where ambient vibration energy is present [8]. The development of self-powered sensors and devices not only reduces environmental hazards related to battery disposals, improves battery lifetime, and in some cases, alleviates the burden of carrying bulky batteries, it also presents new opportunities and challenges in the emerging research field of renewable energy sources for sensors and electronic systems.

The most common sources of vibration energy range from harnessing pressure variations in pipe flow, human walking, flutter of airplane wings, vibration of rotating machinery, packages on conveyer belts, or vehicle motions on bridges, with potential applications to the developments of self-powered sensors for material fatigue detection, structural health monitoring of highway bridges, active damping devices, and surveillance monitoring [4,9–13]. Some feasibility studies have shown that the ambient energy levels in many structural applications experiencing wind and traffic loads are sufficient to power simple wireless sensors [10].

Interests in developing reliable, inexpensive self-powered sensors have led to extensive research in energy harvesting using piezoelectric materials, which when strained, can convert mechanical vibration energy into electrical energy. A comprehensive review of piezoelectric energy harvesting research can be found in Refs. [14] and [15], and surveys and challenges of piezoelectric energy harvesting for MEMS devices are reviewed in Refs. [9], [16], and [17]. It is noted that, while the fundamental designs of vibration energy harvesters involving cantilever beams have not changed much over the years, current research on vibration-based energy harvesting has shifted focus to developing novel techniques to increase the frequency bandwidth of the harvested energy spectrum [18,19]. Some of these techniques include adding proof masses to the cantilever beam and employing arrays of cantilevers and connected beams [3,19–22].

Theoretical analysis of vibration energy harvesters involves the mechanical modeling of the vibrator itself, the modeling of electromechanical coupling via the piezoelectric material, and the electrical modeling of the circuit. The modeling of the mechanical vibrator varies in complexity and technique. Analytical models have taken several different forms, ranging from the simplistic single-degree-of-freedom (SDOF) systems to complex computational methods. In most cases, the mechanical vibrator is modeled as a cantilever beam which may include a proof mass at the tip; however, alternate geometries have also been investigated to increase the power density [23–27]. Comparison of a SDOF model with Euler–Bernoulli predictions shows the inaccuracy of oversimplifying the mechanical modeling of the vibrator [28,29], and modeling using the Timoshenko beam theory [30] confirms the validity of Euler–Bernoulli predictions with the exception of very short beams. In addition to analytical models, computational methods such as finite element methods [31] and equivalent circuit models [32] have been developed for use with harvester circuit analyses and designs.
Of greater interest is the modeling of the coupled electromechanical energy harvester relating the electrical power generated to the vibration input. Basic modeling techniques and power flow analysis for the coupled system have been available since the 1990s [33]. These coupled models can be used to evaluate the effectiveness of the energy conversion of the entire system and thus the performance of the energy harvesters [34−37]. Besides modeling, theoretical predictions of power outputs from prototype vibrators have been verified experimentally, including both macro scale cantilever beams with [38] and without a proof mass [39,40] and MEMS devices [11,41]. These experimental results show that vibration harvesters, which are properly tuned to the input frequency, perform in a manner close to that predicted by conventional modeling.

Although SDOF modeling of the vibration energy harvesters provides a simple, closed-form solution that may be useful for design purposes, it was shown that significant errors could result [28]. A modal analysis method was developed to analyze the response of a distributed parameter piezoelectric cantilever beam model to arbitrary base excitations [42]. This approach, while very useful and commonly adopted, has several drawbacks. First, the response is expressed by an infinite series of eigenfunctions (usually those of beams with classical boundary conditions or an approximate, simpler structure), and these eigenfunctions may be difficult to obtain, particularly for systems with nonproportional damping (nonsymmetric systems). Second, the method does not extend to analyses and designs of beam energy harvesters subjected to nonclassical boundary conditions and intermediate constraints. Such harvester systems may provide new opportunities for optimized designs via frequency tuning. Third, truncation of the series is made in numerical computations. This, along with inaccuracy in the estimation of the eigenfunctions, can result in large numerical errors, especially when the frequency is near the system resonances and antiresonances. Fourth, the series truncation effectively reduces the order of the model, which can lead to loss of information about the higher-order dynamics which may be important for the purposes of control and design of the energy harvesters. It should be noted that most of the modeling techniques have utilized time domain approaches, focusing on the evaluation of the steady-state response. While these time domain methods appear to be effective, the analysis can become quite complex, even for simple geometries of an unimorph cantilever beam, and requires numerous assumptions, such as damping proportionality, in order for the solution to be obtained.

Since the design of beam energy harvesters has focused on the evaluation, both analytically and experimentally, of the frequency response functions (FRFs), it would be desirable to develop analysis techniques in the frequency domain. Response solution of a beam energy harvester in the frequency domain using the transfer function method (TFM) was first proposed in Ref. [43], and later adopted for the free vibration analysis of thick piezoelectric beams and adaptive structures [44,45]. Recently, a Fourier transform Green’s function method (FTGF) was used to derive closed-form expressions for the FRF of the beam response to a harmonic base excitation and the voltage generated by the piezoelectric layer [46]. This method allows for frequency-dependent material properties and damping coefficients to be considered. Their results were verified experimentally and compared well with modal analysis solutions. While these exact derivations are impressive, it is noted that the success of the FTGF technique hinges on the ability to derive the Green’s function of the mechanical beam. Such a derivation becomes very difficult if the beam model does not possess classical boundary conditions or has intermediate constraints. Moreover, the FTGF method is problem specific, i.e., the Green’s functions must be rederived for each problem, and hence does not lead to a systematic approach and a tool for optimized designs of beam harvesters of vibration energy under general boundary conditions and configurations. It should also be noted that, exact closed-form formulas of the FRF are generally not needed in the design of beam energy harvesters; accurate numerical or semi-analytical solutions are sufficient.

In this paper, a systematic approach is proposed to investigate the response and design of vibration energy harvesters based on the concept of distributed transfer functions. The transfer function of a distributed parameter system contains all of the information required to predict the system spectrum, the response under any initial and external disturbances, and the stability of the system response [47]. This formulation has several important advantages over classical time domain methods: (i) it is amenable to algorithmic numerical coding, i.e., the algorithm does not need to be modified when the field equations and boundary conditions are changed [48]; (ii) it can be employed to model nonsymmetric energy harvesters (e.g., nonproportional damping), which have intermediate constraints and are subjected to arbitrary excitations; (iii) the system FRF and its derivatives (such as strains) can be obtained simultaneously from the calculations; and (iv) the formulation leads to compact formulas [49], allowing well-known feedback control principles to be incorporated for optimized design purposes. This paper is organized as follows: Theoretical background on the derivation of the transfer function and FRF of a distributed parameter system, and the computational procedures are first described. The electromechanical coupling is then formulated as a closed-loop feedback. Efficacy of the proposed methodology is demonstrated by applications to the modeling of vibration energy harvesters with nonclassical boundary conditions and intermediate constraints.

2 Theoretical Background

2.1 Frequency Response Function of a Distributed Parameter System. In this section, basic results for the derivation of the transfer function of a distributed parameter model are outlined; details are found in Ref. [48]. Consider the response \( w(x,t) \) of a one-dimensional distributed parameter element subjected to an excitation \( f(x,t) \)

\[
\left( A \frac{\partial^2}{\partial x^2} + B \frac{\partial}{\partial t} + C \right) w(x,t) = f(x,t), \quad x \in (0, L), \quad t > 0 \tag{1}
\]

with inhomogeneous boundary conditions (such as base excitations for vibration harvesters)

\[
M_j w(x,t) \big|_{x=0} + N_j w(x,t) \big|_{x=L} = \gamma_j(t), \quad j = 1, 2, ..., n \tag{2}
\]

where \( M_j \) and \( N_j \) are temporal-spatial, linear differential operators of the proper order, and \( \gamma_j(t) \) are known functions that represent boundary inhomogeneity. In Eq. (1), \( A, B, \) and \( C \) are spatial differential operators of the form

\[
A = \sum_{k=0}^{n} a_k \frac{\partial^k}{\partial x^k}, \quad B = \sum_{k=0}^{n} b_k \frac{\partial^k}{\partial x^k}, \quad C = \sum_{k=0}^{n} c_k \frac{\partial^k}{\partial x^k} \tag{3}
\]

which are associated with the inertia; damping, Coriolis acceleration and mass transport; and stiffness, centrifugal forces, and circulatory effects, respectively. The initial conditions are

\[
w(x,t) \big|_{t=0} = u_0(x), \quad \frac{\partial w(x,t)}{\partial t} \big|_{t=0} = v_0(x), \quad x \in (0, L) \tag{4}
\]

Laplace transform of Eq. (1) to the frequency domain leads to
\[(s^2 A + sB + C) \mathbf{w}(x, s) = \mathbf{f}(x, s)\]
\[\equiv \tilde{f}(x, s) + (sA + B)u_0(x) + Av_0(x) \quad (5)\]

where \(\mathbf{w}(x, s)\) and \(\tilde{f}(x, s)\) are the Laplace transforms of \(w(x, t)\) and \(f(x, t)\), respectively, and \(s\) is the complex frequency parameter. Laplace transform of the boundary conditions (2) gives

\[\mathbf{M} \mathbf{w}(x, s) |_{x=0} + N \mathbf{w}(x, s) |_{x=L} = \tilde{\gamma}_j(s) \equiv \tilde{\gamma}_{Bj}(s) \quad j = 1, 2, \ldots, n \quad (6)\]

where \(\mathbf{M}\) and \(\mathbf{N}\) are the operators \(M_j\) and \(N_j\) with the time-derivative operators \((\partial/\partial t)\) and \((\partial^2 / \partial t^2)\) replaced by \(s\) and \(s^2\), respectively, \(\tilde{\gamma}_{Bj}(s)\) is the Laplace transform of \(\gamma_j(t)\), and \(\tilde{\gamma}_j(s)\) is a polynomial of \(s\) representing the initial conditions at the boundaries \(x = 0\) and \(L\) respectively. Symbolically, the solution for Eqs. (5) and (6) can be written in the form [50]

\[\mathbf{w}(x, s) = \int_0^L \left[ \mathbf{W}_o(x, \xi, s) \tilde{\gamma}_j(s) \right] d\xi + \sum_{j=1}^n \phi_j(x, s) \tilde{\gamma}_j(s) \quad (7)\]

where the integral kernel, \(\mathbf{W}_o(x, \xi, s)\), is the transfer function of the distributed parameter system (1), and the boundary transfer functions \(\phi_j(x, s)\) represent the influence of the inhomogeneity of the boundary conditions (6) on the system response. It is noted that the inverse Laplace transfer of \(\mathbf{W}_o(x, \xi, s)\) is the Green’s function of the distributed parameter system (1).

To obtain the transfer function of the distributed parameter system, first cast the problem in the state-space form

\[\frac{\partial}{\partial x} \mathbf{\eta}(x, s) = \mathbf{F}(s) \mathbf{\eta}(x, s) + \mathbf{q}(x, s), \quad x \in (0, L) \]

\[\mathbf{M}(s) \mathbf{\eta}(0, s) + \mathbf{N}(s) \mathbf{\eta}(L, s) = \gamma(s) \quad (8)\]

where the state vector \(\mathbf{\eta}(x, s)\) and other variables are defined as:

\[\mathbf{\eta}(x, s) = \left\{ \begin{array}{c} \mathbf{w}(x, s) \\ \frac{\partial}{\partial x} \mathbf{w}(x, s) \\ \vdots \\ \frac{\partial^{n-1}}{\partial x^{n-1}} \mathbf{w}(x, s) \end{array} \right\} \in C^n \]

\[\mathbf{q}(x, s) = \left\{ \begin{array}{c} \mathbf{f}(x, s) \\ 0 \\ \vdots \\ 0 \\ \frac{\partial}{\partial x} \mathbf{f}(x, s) \end{array} \right\} \in C^n \]

\[\gamma(s) = \left\{ \begin{array}{c} \tilde{\gamma}_1(s) \\ \tilde{\gamma}_2(s) \\ \vdots \\ \tilde{\gamma}_n(s) \end{array} \right\} \in C^n \]

\[\mathbf{F}(s) = \left[ \begin{array}{cccc} 0 & 1 & & \vdots \\ -d_0(s) & -d_1(s) & \cdots & -d_{a-2}(s) & -d_{a-1}(s) \end{array} \right] \quad \in C^{a}\times\{a\} \quad (9)\]

where the coefficients of the matrix \(\mathbf{F}(s)\) are

\[d_k(s) = \frac{a_k s^2 + b_k s + c_k}{a_n s^2 + b_n s + c_n}, \quad k = 0, 1, \ldots, n-1\]

and \(\mathbf{M}(s)\) and \(\mathbf{N}(s)\) are \(n \times n\) complex matrices, consisting of the coefficients of the operators \(\mathbf{M}_j\) and \(\mathbf{N}_j\), \(j = 1, 2, \ldots, n\). Note that, for uniformly distributed parameter elements, \(\mathbf{F}(s)\) is not a function of \(x\). A compact formula can be derived for the solution of this problem

\[\mathbf{\eta}(x, s) = \int_0^L \mathbf{G}(x, \xi, s) \mathbf{q}(\xi, s) d\xi + \mathbf{H}(x, s) \gamma(s), \quad x \in (0, L) \quad (10)\]

where the matrix Green’s function written in terms of the fundamental matrix, \(\mathbf{\Phi}(x, s) = e^{\mathbf{H}(x, s)}\), is

\[\mathbf{G}(x, \xi, s) = \left\{ \begin{array}{c} \mathbf{\Phi}(x, s)(\mathbf{M}(s) + \mathbf{N}(s)\mathbf{\Phi}(L, s))^{-1}\mathbf{M}(s) \mathbf{\Phi}(-\xi, s), \xi < x \\ -\mathbf{\Phi}(x, s)(\mathbf{M}(s) + \mathbf{N}(s)\mathbf{\Phi}(L, s))^{-1}\mathbf{N}(s) \mathbf{\Phi}(L - \xi, s), \xi > x \end{array} \right\} \quad (11)\]

By Eqs. (9)–(13), it is seen that

\[\mathbf{w}(x, s) = \int_0^L \frac{g_{kn}(x, \xi, s)}{a_n s^2 + b_n s + c_n} \tilde{\gamma}_j(s) d\xi + \sum_{j=1}^n h_{ij}(x, s) \tilde{\gamma}_j(s) \quad (14)\]

The distributed transfer function and the boundary transfer functions of the system are thus given by

\[W_o(x, \xi, s) = \frac{g_{kn}(x, \xi, s)}{a_n s^2 + b_n s + c_n}, \quad \phi_j(x, s) = h_{ij}(x, s), \quad j = 1, 2, \ldots, n \quad (15)\]

2.2 Computations of the System Response. Equations (11), (12), and (15) suggest a new method for evaluation of the system transfer function. To determine \(W_o(x, \xi, s)\) and \(\phi_j(x, s)\), one only needs to calculate the fundamental matrix \(\mathbf{\Phi}(x, s) = e^{\mathbf{H}(x, s)}\) and
(M(x) + N(x) Φ(L, s))^{-1}. These two matrices, with low-order in general (for example, for vibration and wave propagation in continuous, usually n ≤ 4) can be accurately calculated. Note that G(x, ξ, s) and H(x, s) in Eqs. (11) and (12) are expressed in closed-form. Therefore, W(x, ξ, s) and Φ(x, s) can be precisely evaluated in closed-form. The numerical scheme for the evaluation of W(x, ξ, s) and Φ(x, s) is outlined as follows:

(i) For a given system (1) and boundary conditions (2), form the matrices F(s), M(s), and N(s) which are very simple to model.

(ii) Given x and s, evaluate the matrices e^{-xF(s)} and (M(s) + N(s)) e^{-xF(s)}^{-1}.

(iii) Calculate W(x, ξ, s) and Φ(x, s) based on Eqs. (11), (12), and (15).

Remarks on the computations and evaluations of the system response and derivatives (e.g., strain) for energy harvester designs:

1. There are many methods that give exact presentations of Φ(x, s) ([51, 52]). However, because of the special form of F(s), matrix diagonalization may be simpler to use. With advances in computational tools such as MATLAB, these computations can be executed with ease and precision. The incorporation of control principles using MATLAB toolboxes makes this methodology amenable to numerical coding for more complex problems in the designs of vibration energy harvesters.

2. The proposed method is exact; no truncation or approximation has been made. It requires no knowledge of the system eigensolutions, and the method is valid for both self-adjoint and non-self-adjoint systems. Response of the beam and its higher-order derivatives (e.g., strains) are obtained simultaneously, without performing any differentiation, through the system transfer functions expressed in Eq. (13).

3. The aforementioned numerical scheme is convenient in computer coding. The differential equations and the boundary conditions are defined through easy assignments of the matrices F(s), M(s), and N(s). Because F(s) is independent of M(s) and N(s), changes in the boundary conditions do not require re-calculation of the fundamental matrix e^{-xF(s)}. Indeed, changes in boundary conditions require only minor changes in the definitions of M(s) and N(s); no modification in the computation algorithms is needed. The proposed methodology can, thus, treat systems with complicated boundary conditions.

4. According to Eqs. (11) and (12), the characteristic equation of the distributed parameter system (1) is

\[ \det(M(x) + N(x) Φ(x, s)) = 0 \]  

where its roots are the eigenvalues of the distributed parameter system, or the poles of W(x, ξ, s) and Φ(x, s).

5. Although this paper is focused on application to distributed piezoelectric models with constant parameters, the approach discussed can be extended to distributed systems with parameters as functions of the spatial coordinate x, i.e., a₀ = a₀(x), b₀ = b₀(x), and c₀ = c₀(x). In this case, the matrix formulation for W(x, ξ, s) and Φ(x, s) is still valid in terms of a fundamental matrix Φ(x, s). For general distributed parameter systems, closed-form expressions for Φ(x, s) are usually difficult to obtain, although approximation methods are available [53]. However, for a non-uniformly distributed parameter system, exact determination of its eigensolutions is also difficult (and may be more difficult). So, the eigenfunction expansion series representation of the system transfer function may not be appropriate at all.

6. In the evaluation of the transfer functions, there is no need to derive the characteristic equation, which may be very tedious for systems with nonclassical, complex boundary conditions.

The transfer function method can effectively formulate coupled electromechanical problems using a closed-loop feedback control principle (treating the voltage, which depends on the beam deflection, as feedback) which will be discussed later in this paper.

It is noted that the frequency response of the distributed parameter model can easily be obtained. Consider the case of homogeneous boundary conditions and a harmonic point force of unit amplitude applied at ξ = ξ₀. The closed-form steady-state response is obtained by simply replacing s by jω in Eq. (14)

\[ w(x, t) = W_ν(x, ξ_ν, Ω)e^{jωt} \]

General formulas and procedures for evaluating the transient response can be found in Ref. [54]. The evaluation of the mode shapes and strain distribution can also be obtained by solving the associated eigenvalues problem [55].

3 Piezoelectric Energy Harvester model

3.1 Mechanical Beam Model. The proposed frequency domain method is applied to a distributed parameter model of unimorph vibration energy harvester described in Ref. [42]. The model is a uniform composite Euler–Bernoulli beam consisting of a lead zirconate titanate (PZT) layer bonded to the substrate layer, where the electrical feedback from the harvesting circuit is not considered in the mechanical modeling. The boundary conditions are standard for a cantilever beam with harmonic base excitations (translation with small rotation) at one end, see Fig. 1.

There are two different formulations to this basic mechanical modeling problem: modeling the base excitations as external forces or as boundary conditions. It will be shown that, while modeling the base excitations as boundary conditions would usually make classical approaches much more complicated, the transfer function method leads to more straightforward formulation and solution.

3.1.1 Base Excitations Modeled as External Forces. The governing equation of motion of the uncoupled beam can be expressed as

\[ m \frac{d^2w(x,t)}{dx^2} + c_a \frac{dw(x,t)}{dt} + c_e \frac{d^2w(x,t)}{dx^2} + EI \frac{d^4w(x,t)}{dx^4} = -m \frac{d^2w_b(x,t)}{dx^2} - c_a \frac{dw_b(x,t)}{dt} \]

where \(w_b(x,t)\) is the base excitation and \(w(x,t)\) is the response of the beam relative to the base excitation; the total displacement is \(W(x,t) = w_b(x,t) + w(x,t)\). The parameters are defined as \(m\) the mass per unit length, \(c_e\) the equivalent strain rate damping, \(c_a\) the viscous air damping coefficient, and \(EI\) the product of the Young’s modulus and moment of inertia.

Fig. 1 Schematic of a unimorph vibration energy harvester under base excitations
The bending stiffness of the composite cross section EI can be expressed as:

$$EI = b \left[ E_s (h_s^3 - h_a^3) + E_p (h_p^3 - h_b^3) \right]$$

where \( b \) is the width of the beam, \( E_s \) and \( E_p \) are the Young’s modulus of the substrate and piezoelectric layer, respectively, \( h_s \) and \( h_b \) are the distances from the neutral axis to the bottom and top of the substrate layer, respectively, and \( h_p \) is the distance from the top of the piezoelectric layer to the neutral axis. For an unimorph piezoelectric beam, \( m \) is defined as:

$$m = b (h_p \rho_s + h_b \rho_p)$$

where \( h_p \) and \( h_b \) are the thickness of the substrate and piezoelectric layer, respectively, and \( \rho_s \) and \( \rho_p \) are the density of the substrate and piezoelectric layer, respectively.

In Fig. 1, \( y(t) \) and \( \theta(t) \) are the linear and angular harmonic boundary excitations applied at \( x = 0 \) and the base excitation is, thus, expressed as \( w_b(x,t) = (y_0 e^{i\omega t} + x \theta_0 e^{i\omega t}) \delta(x - 0) \), where \( \delta(\cdot) \) is the Dirac-Delta function. The boundary conditions for the equations of motion are

\[
\begin{align*}
  w(x, t) \bigg|_{x=0} &= 0, & \frac{\partial w(x, t)}{\partial x} \bigg|_{x=0} &= 0, & \frac{\partial^2 w(x, t)}{\partial x^2} \bigg|_{x=L} &= 0, \\
  \frac{\partial^3 w(x, t)}{\partial x^3} \bigg|_{x=L} &= 0.
\end{align*}
\]

Performing the Laplace transform gives

\[
\begin{align*}
  (ms^2 + c_\omega s^2) \overline{w}(s, s) + (c_i s + EI) \frac{\partial^2 \overline{w}(s, s)}{\partial x^2} &= (-ms^2 - c_\omega s^2) \overline{w}_b(s, s), \tag{20} \\
  \overline{w}(s, s) \bigg|_{x=0} &= 0, & \frac{\partial \overline{w}(s, s)}{\partial x} \bigg|_{x=0} &= 0, & \frac{\partial^2 \overline{w}(s, s)}{\partial x^2} \bigg|_{x=L} &= 0, \\
  \frac{\partial^3 \overline{w}(s, s)}{\partial x^3} \bigg|_{x=L} &= 0. \tag{21}
\end{align*}
\]

Substituting Eqs. (20) and (21) in the state-space formulation of Eq. (9) results in

\[
\begin{align*}
  F(s) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
    ms^2 + c_\omega s^2 & c_i s + EI & 0 & 0 \end{bmatrix}, & M(s) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \end{bmatrix}, \\
  N(s) &= \begin{bmatrix} 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1 \end{bmatrix}, & \gamma(s) &= \begin{bmatrix} 0 \\
    0 \\
    0 \\
    0 \end{bmatrix},
\end{align*}
\]

(22)

With the earlier definitions, the distributed transfer function of the frequency response of the piezoelectric energy harvester model can be obtained using the computation procedure outlined in Sec. 2. In particular, once the system transfer function is obtained, Eq. (17) can be used to evaluate the system response to harmonic input excitations, i.e., the FRF solution. The harmonic steady-state response is then given by Eq. (14)

$$\overline{w}(x, s) = \int_0^L \frac{g_{14}(\xi, \xi, s)}{c_i s + EI} \overline{F}(\xi, s) d\xi$$

(23)

where

$$\overline{F}(x, s) = \begin{bmatrix} f(x) \\
    \cdots \\
    f(x) \end{bmatrix}$$

(24)

and

$$\overline{w}(x, s) = \begin{bmatrix} y(x) \\
    \cdots \\
    y(x) \end{bmatrix}$$

(25)

with \( y(x) = y_0 e^{i\omega t} \), \( \theta(t) = \theta_0 e^{i\omega t} \).

### 3.1.2 Base Excitations Modeled as Boundary Conditions

An alternate technique is to model the base excitations as inhomogeneous boundary conditions rather than external excitations. In this technique, the total displacement \( W(x, t) \) of the beam is used and the beam equation and boundary conditions become

\[
\begin{align*}
  &m \frac{\partial^2 W(x, t)}{\partial x^2} + c_a \frac{\partial W(x, t)}{\partial t} + c_i \frac{\partial^2 W(x, t)}{\partial x^2 \partial t} + EI \frac{\partial^4 W(x, t)}{\partial x^4} = 0, \\
  &W(x, t) \bigg|_{x=0} = y(t), & \frac{\partial W(x, t)}{\partial x} \bigg|_{x=0} = \theta(t), \\
  &\frac{\partial^2 W(x, t)}{\partial x^2} \bigg|_{x=L} = 0, & \frac{\partial^3 W(x, t)}{\partial x^3} \bigg|_{x=L} = 0. \tag{25}
\end{align*}
\]

Taking the Laplace transform of Eq. (25) gives

\[
\begin{align*}
  &\left(ms^2 + c_\omega s^2\right) \overline{W}(s, s) + (c_i s + EI) \frac{\partial^2 \overline{W}(s, s)}{\partial x^2} = 0, \\
  &\overline{W}(s, s) \bigg|_{x=0} = \overline{y}(s), & \frac{\partial \overline{W}(s, s)}{\partial x} \bigg|_{x=0} = \overline{\theta}(s), \\
  &\frac{\partial^2 \overline{W}(s, s)}{\partial x^2} \bigg|_{x=L} = 0, & \frac{\partial^3 \overline{W}(s, s)}{\partial x^3} \bigg|_{x=L} = 0. \tag{26}
\end{align*}
\]

When this approach is employed, the boundary conditions in the state-space formulation are modified to include the excitation at the boundary

$$\gamma(s) = \begin{bmatrix} y_0 \theta_0 \theta_0 \theta_0 \end{bmatrix}^T$$

and the forcing function \( \overline{F}(x, s) \) is reduced to zero. This formulation results in a much simpler solution procedure to the stated problem and produces the same solutions as those found using the previous formulation. The harmonic steady-state response is then given by Eq. (14):

$$\overline{W}(x, s) = [h_{11}(x, s) y_0 + h_{12}(x, s) \theta_0$$

and the relative displacement of the beam, \( w(x, t) = W(x, t) - w_b(x, t) \), is thus

$$\overline{w}(x, s) = [h_{11}(x, s) - 1]y_0 + [h_{12}(x, s) - x] \theta_0$$

(29)

For the case of \( \theta_0 = 0 \), \( w(x, t) = [h_{11}(x, c) - 1]y_0 e^{i\omega t} \). Note that either Eq. (23) or Eq. (29) can be implemented to find the relative displacement of the beam. However, Eq. (23) requires a numerical integration which can result in a higher computational cost.

### 3.2 Results and Discussions of Uncoupled Model

For validation and comparison of our results to the literature, the cantilever dimensions and material properties are those listed in Ref. [42], as shown in Table 1.

As shown in Fig. 2(a), the methods are identical except near the antiresonance frequency. Closer investigation of this area, as seen in Fig. 2(b), shows that additional terms in the modal analysis converge to the transfer function method results developed here for the undamped beam model. The formulation presented and the accompanying plots show that the frequency response can be obtained in a straightforward manner using the proposed frequency domain analysis.

Note that the fundamental characteristics of the transfer function model of the cantilever beam can easily be deduced from this
Table 1 Geometrical, material and electromechanical parameters of the unimorph cantilever beam

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of the beam</td>
<td>L = 100 mm</td>
</tr>
<tr>
<td>Width of the beam</td>
<td>b = 20 mm</td>
</tr>
<tr>
<td>PZT electrode start distance from the base</td>
<td>x₁ = 0 mm</td>
</tr>
<tr>
<td>PZT electrode end distance from the base</td>
<td>x₂ = 100 mm</td>
</tr>
<tr>
<td>Thickness of the substrate layer</td>
<td>hᵣ = 0.5 mm</td>
</tr>
<tr>
<td>Thickness of the piezoelectric layer</td>
<td>hₚ = 0.4 mm</td>
</tr>
<tr>
<td>Young’s modulus of the substrate layer</td>
<td>Eᵣ = 100 GPa</td>
</tr>
<tr>
<td>Young’s modulus of the piezoelectric layer</td>
<td>Eₚ = 66 GPa</td>
</tr>
<tr>
<td>Equivalent damping due to structural viscoelasticity</td>
<td>cᵣ/Eᵣ = 1.2433 × 10⁻⁵ s/rad</td>
</tr>
<tr>
<td>Damping due to air</td>
<td>cₐ/m = 4.866 rad/s</td>
</tr>
<tr>
<td>Density of the substrate layer</td>
<td>ρᵣ = 7165 kg/m³</td>
</tr>
<tr>
<td>Density of the piezoelectric layer</td>
<td>ρₚ = 7800 kg/m³</td>
</tr>
<tr>
<td>Piezoelectric constant</td>
<td>d₃₁ = -190 pm/V</td>
</tr>
<tr>
<td>Permittivity</td>
<td>eᵣ = 15.93 nF/m</td>
</tr>
</tbody>
</table>
The state and boundary matrices $F(s)$, $M(s)$, $N(s)$, and $\gamma(s)$ of the state-space formulation remain identical to those used in the uncoupled mechanical system, Eq. (22), and as before, the beam response is

$$\mathbf{w}(x, s) = \int_0^L \frac{g_{13}(x, \xi, s)}{c_4 J_s + EI} \mathcal{F}_1(\xi, s) \, d\xi$$

(32)

The forcing function contains the electrode voltage term which provides the electromechanical coupling as such

$$\mathcal{F}_1(x, s) = \mathcal{F}(x, s) + \vartheta \tau (s) \times \left[ \frac{d\delta(x - x_1)}{dx} - \frac{d\delta(x - x_2)}{dx} \right]$$

(33)

where $\mathcal{F}(x, s)$ is the forcing function for the uncoupled problem, shown in Eq. (24). The resulting beam response is

$$\mathbf{w}(x, s) = \int_0^L \frac{g_{13}(x, \xi, s)}{c_4 J_s + EI} \left[ \mathcal{F}(\xi, s) + \vartheta \tau (s) \times \left[ \frac{d\delta(\xi - x_2)}{d\xi} - \frac{d\delta(\xi - x_1)}{d\xi} \right] \right] d\xi$$

(34)

Next, the equation for the voltage $v(t)$ is investigated. Based on the piezoelectric constitutive equations, the relationship between the beam displacement and the voltage output can be stated as

$$\frac{\varepsilon_{13}^2 b (x_2 - x_1)}{h_0} \frac{dv(t)}{dt} + v(t) = -\int_{x_1}^{x_2} d_n Y_p h_p b \frac{\partial^3 \mathbf{w}(x, t)}{\partial x^3 \partial t} \, dx$$

(35)

where $\varepsilon_{13}$ is the permittivity at constant stress and $R_l$ is the electric resistive load.

Taking the Laplace transform and assuming zero initial conditions result in

$$\left( \frac{\varepsilon_{13}^2 b (x_2 - x_1)}{h_0} \right) \mathcal{F}(s) + \frac{1}{R_l} \mathcal{V}(s) = -\int_{x_1}^{x_2} d_n Y_p h_p b s \frac{\partial^2 \mathbf{w}(x, s)}{\partial x^2} \, dx$$

(36)

Applying integration by parts to the integral and solve for the voltage

$$\mathcal{V}(s) = \frac{\varphi_0}{\varphi_0 + 1} \left[ \frac{\partial \mathcal{F}(x, s)}{\partial x} \right]_{x=x_1} - \frac{\partial \mathcal{F}(x, s)}{\partial x} \bigg|_{x=x_2}$$

(37)

where $\varphi_0 = -R_l d_{13} Y_p h_p b$, and $\tau_0 = (R_l \varepsilon_{13}^2 b (x_2 - x_1)) / h_0$. The relationship between the voltage and the strain is clearly defined. This relationship is very important since it yields the equivalent mechanical terms for the electrical coupling. Based on Eqs. (34) and (37), the voltage term $\vartheta \mathcal{F}(s)$ can be viewed as a feedback force, which depends on the strains, to the beam response. The voltage can be eliminated from the coupled equations by substituting the voltage from Eqs. (37) into (34)

$$\mathbf{w}(x, s) = \frac{1}{c_4 J_s + EI} \int_0^L \frac{g_{13}(x, \xi, s)}{c_4 J_s + EI} \left[ \mathcal{F}(\xi, s) + \frac{\partial \varphi_0}{\tau_0 + 1} \left[ \frac{\partial \mathcal{F}(x, s)}{\partial x} \right]_{x=x_1} - \frac{\partial \mathcal{F}(x, s)}{\partial x} \bigg|_{x=x_2} \right] \times \left[ \frac{d\delta(\xi - x_2)}{d\xi} - \frac{d\delta(\xi - x_1)}{d\xi} \right] d\xi$$

(38)

This equation can be further simplified by using the following Dirac delta function property:

$$\int_{-\infty}^{\infty} d^2 \delta(x - x_0) f(x) \, dx = (-1)^n \frac{d^n f(x_0)}{dx^n}$$

and the Green’s function property from [56]

$$\left[ \frac{\partial \mathbf{G}(x, \xi, s)}{\partial \xi} \right]_{x_0} = -g_{1n-1}(x, \xi, s) + \kappa_{0-1}(s) g_{1n}(x, \xi, s)$$

(39)

where $g_{1n-1}(x, \xi, s)$ denotes the $(1, n - 1)$ -th element of $\mathbf{G}(x, \xi, s)$, and $\kappa_k(s)$ are defined as

$$\kappa_0(s) = \frac{x^2 + a_0 s + b_0}{a_0 s + b_0}, \quad \kappa_k(s) = \frac{a_k s + b_k}{a_0 s + b_0}, \quad k = 1, \ldots, n - 1$$

Hence, Eq. (38) becomes

$$\mathbf{w}(x, s) = \int_0^L \frac{g_{13}(x, \xi, s)}{c_4 J_s + EI} \mathcal{F}(\xi, s) d\xi + \frac{\partial \varphi_0}{\tau_0 + 1} \frac{\partial \mathcal{F}(x_2, s)}{\partial x} \bigg|_{x=x_1} \left[ g_{1n}(x, x_2, s) - g_{13}(x, x_1, s) \right]$$

(40)

The bending slope of the beam ($\partial \mathcal{F}/\partial x$) at any given point $x$ can be calculated by taking the spatial derivative of the beam displacement given by Eq. (40) and employing the following relationship for the spatial derivative of the transfer function [56]:

$$\left[ \frac{\partial \mathbf{G}(x, \xi, s)}{\partial \xi} \right]_{x_0} = g_{2n}(x, \xi, s)$$

(41)
Hence, the bending slope can be expressed as

\[
\frac{\partial \bar{\varpi}(x,s)}{\partial x} = 2 \int_0^t g_{23}(x,\zeta,s) \bar{f}(\zeta) d\zeta + \frac{\partial \varphi_s}{\partial x} \left[ \frac{\partial \bar{\varpi}(x_2,s)}{\partial x} - \frac{\partial \bar{\varpi}(x_1,s)}{\partial x} \right] \left[ g_{23}(x,x_2,s) - g_{23}(x,x_1,s) \right]
\]  

(42)

The feedback approach can now be applied by substituting \( x = x_1 \) and \( x_2 \) into Eq. (42) and solving for unknowns \( \left. \frac{\partial \bar{\varpi}(x,s)}{\partial x} \right|_{x=x_1} \) and \( \left. \frac{\partial \bar{\varpi}(x,s)}{\partial x} \right|_{x=x_2} \) as follows:

\[
\left. \frac{\partial \bar{\varpi}(x,s)}{\partial x} \right|_{x=x_1} = A_1 B_2 + B_1 - A_2 B_1
\]

(43)

where the symbols on the right-hand side of the aforementioned equation are defined as

\[
A_i = \frac{\partial \varphi_s}{(\tau_s + 1)(c_is + EI)} \left[ g_{23}(x_i,x_2,s) - g_{23}(x_i,x_1,s) \right], \quad i = 1, 2
\]

(44)

\[
B_i = \int_0^t \frac{g_{24}(x_i,\zeta,s)}{c_is + EI} \bar{f}(\zeta) d\zeta, \quad i = 1, 2
\]

(45)

Using Eq. (43), the value of \( \left. \frac{\partial \bar{\varpi}(x,s)}{\partial x} \right|_{x=x_2} - \left. \frac{\partial \bar{\varpi}(x,s)}{\partial x} \right|_{x=x_1} \) can be written as

\[
\left. \frac{\partial \bar{\varpi}(x,s)}{\partial x} \right|_{x=x_2} - \left. \frac{\partial \bar{\varpi}(x,s)}{\partial x} \right|_{x=x_1} = \int_0^t \frac{g_{24}(x_2,\zeta,s) - g_{24}(x_1,\zeta,s)}{c_is + EI} \bar{f}(\zeta) d\zeta
\]

(46)

The harmonic steady-state response is then obtained by substituting Eq. (46) into Eq. (40)

\[
\bar{w}(x,s) = \frac{\partial \varphi_s}{(\tau_2 + 1)(c_is + EI)} \left[ g_{23}(x_1,x_2,s) - g_{23}(x_1,x_1,s) + g_{23}(x_2,x_1,s) - g_{23}(x_2,x_2,s) \right]
\]

(47)

\[
3.3.2 \text{ Base Excitations Modeled as Boundary Conditions.} \text{ The governing equation of motion becomes}
\]

\[
m \frac{\partial^2 \bar{w}(x,t)}{\partial t^2} + c_a \frac{\partial \bar{w}(x,t)}{\partial t} + c_i \frac{\partial^2 \bar{w}(x,t)}{\partial x^2} + EI \frac{\partial^4 \bar{w}(x,t)}{\partial x^4} = \partial \varphi(t) \times \left[ \frac{\partial \bar{w}(x_2,s)}{\partial x} - \frac{\partial \bar{w}(x_1,s)}{\partial x} \right]
\]

(48)

where \( \bar{w}(x,t) \) is the absolute deflection of the beam. Laplace transform of Eq. (48) gives:

\[
(m^2 + c_0) \bar{W}(x,s) + (c_is + EI) \frac{\partial^2 \bar{W}(x,s)}{\partial x^2} = \partial \varphi(s) \times \left[ \frac{\partial \bar{w}(x_2,s)}{\partial x} - \frac{\partial \bar{w}(x_1,s)}{\partial x} \right]
\]

(49)

The boundary conditions are the same as Eq. (26). Thus

\[
q(x,s) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \gamma = \begin{bmatrix} y_0 \\ \dot{y}_0 \\ 0 \\ 0 \end{bmatrix}
\]

(50)

where \( \bar{f}(x,s) \) is given by Eq. (33) with zero external force \( \bar{f}(x,s) = 0 \). Based on the state-space formulation, the steady-state response is given by

\[
\bar{W}(x,s) = \int_0^t g_{14}(x,\zeta,s) \left\{ \partial \varphi(s) \times \left[ \frac{\partial \bar{w}(x_2,s)}{\partial x} - \frac{\partial \bar{w}(x_1,s)}{\partial x} \right] \right\} d\zeta + h_{11}(x,s) \gamma(s) + h_{12}(x,s) \theta(s)
\]

(51)

From Eq. (37), the voltage is expressed in terms of the relative deflection \( \bar{w}(x,s) \), which would make the expression not applicable for the current formulation when the base excitation is treated as a boundary condition. However, it can be shown that the voltage can also...
be expressed in terms of the total displacement of the beam to allow the coupled problem, as defined by Eq. (51), to be solved. Using the notations defined earlier, we have

$$W(x, s) = w(x, s) + y(s) + x \, \overline{u}(s)$$  \hspace{1cm} (52)

Taking the derivative of $W(x, s)$ leads to

$$\frac{\partial W(x, s)}{\partial x} = \frac{\partial w(x, s)}{\partial x} + \frac{\partial y(s)}{\partial x} + \frac{\partial \overline{u}(s)}{\partial x}$$ \hspace{1cm} (53)

Substituting Eq. (53) into Eq. (37), the voltage can be expressed in terms of the total displacement

$$V(s) = \frac{\varphi_s}{\tau_s + 1} \left[ \frac{\partial W(x, s)}{\partial x} \bigg|_{x=x_1} - \frac{\partial W(x, s)}{\partial x} \bigg|_{x=x_2} \right]$$ \hspace{1cm} (54)

Note that, since the voltage depends on the difference of the slopes, the influence of the boundary rigid body rotation $\overline{u}(s)$ does not contribute to the voltage generation. Employing Eq. (41) and substituting the voltage from Eq. (54) into Eq. (51), the response becomes

$$W(x, s) = \varphi_s \left[ \frac{\partial W(x, s)}{\partial x} \bigg|_{x=x_1} - \frac{\partial W(x, s)}{\partial x} \bigg|_{x=x_2} \right] \times \left[ g_{13}(x, x_2, s) - g_{13}(x, x_1, s) \right] + h_{11}(x, s) V(s) + h_{12}(x, s) \overline{u}(s)$$ \hspace{1cm} (55)

The bending slope of the beam $\left( \frac{\partial W(x, s)}{\partial x} \right)$ at any given point $x$ can be obtained by taking the spatial derivative of the beam displacement from Eq. (55). Using this derivative along with Eq. (41) and the following equation for the spatial derivative of the boundary trans-fer function:

$$\left[ \frac{\partial H(x, s)}{\partial x} \right]_{1j} = h_{2j}(x, s) \quad j = 1, 2, 3, \ldots, n$$ \hspace{1cm} (56)

the bending slope of the beam becomes

$$\frac{\partial W(x, s)}{\partial x} = \frac{\varphi_s}{\tau_s + 1} \left[ \frac{\partial W(x, s)}{\partial x} \bigg|_{x=x_1} - \frac{\partial W(x, s)}{\partial x} \bigg|_{x=x_2} \right] \times \left[ g_{23}(x, x_2, s) - g_{23}(x, x_1, s) \right] + h_{21}(x, s) V(s) + h_{22}(x, s) \overline{u}(s)$$ \hspace{1cm} (57)

Substituting $x = x_1$ and $x_2$ into Eq. (57) to solve for the bending slopes

$$\left. \frac{\partial W(x, s)}{\partial x} \right|_{x=x_1} = \frac{A_1 C_2 + C_i - A_2 C_i}{1 + A_1 - A_2}, \quad i = 1, 2$$ \hspace{1cm} (58)

where

$$C_i = h_{21}(x_i, s) V(s) + h_{22}(x_i, s) \overline{u}(s)$$ \hspace{1cm} (59)

From Eq. (58), the bending slope difference can be written as

$$\left. \frac{\partial W(x, s)}{\partial x} \bigg|_{x=x_2} - \frac{\partial W(x, s)}{\partial x} \bigg|_{x=x_1} \right] = \frac{\left[ h_{21}(x_2, s) - h_{21}(x_1, s) \right] V(s) + \left[ h_{22}(x_2, s) - h_{22}(x_1, s) \right] \overline{u}(s)}{1 + \varphi_s \left[ \frac{\tau_s + 1}{(c_s Is) \overline{u} + EI} \right] \left[ g_{23}(x_1, x_2, s) - g_{23}(x_1, x_1, s) + g_{23}(x_2, x_1, s) - g_{23}(x_2, x_2, s) \right]}$$ \hspace{1cm} (60)

The harmonic steady-state response is then obtained by substituting Eq. (60) into Eq. (55)

$$W(x, s) = \frac{\varphi_s}{\tau_s + 1} \left[ \frac{\partial W(x, s)}{\partial x} \bigg|_{x=x_1} - \frac{\partial W(x, s)}{\partial x} \bigg|_{x=x_2} \right] \times \left[ g_{13}(x, x_2, s) - g_{13}(x, x_1, s) \right] + h_{11}(x, s) V(s) + h_{12}(x, s) \overline{u}(s)$$ \hspace{1cm} (61)

The harmonic steady-state response for the relative displacement of the beam is then obtained by substituting Eq. (61) into Eq. (52).

### 3.4 Validation and Discussion of the Transfer Function Method Solution for the Coupled Electromechanical Problem

Transfer function method solution for the coupled piezoelectric cantilever is validated against the modal analysis solution [42]. Since the exact solution is obtained by the TFM, with no mode truncation, the modal analysis solution is, thus, expected to converge to the TFM solution as the number of modes used in the eigenfunction expansion increases. Figure 3 compares the TFM solution with the modal analysis results with 3 modes and 10 modes. The agreement is excellent.

We have demonstrated the application of the transfer function method to the coupled unimorph piezoelectric cantilever energy harvester problem. The proposed method leads to a powerful and systematic computational tool for the analysis and design of electromechanical
systems, even with intermediate constraints. Unlike the modal analysis approach [42], the TFM does not require the determination of the eigenfunctions, which may be difficult to obtain for beams with nonclassical boundary conditions. In Sec. 4, examples of beam harvesters with nonclassical boundary conditions and with intermediate constraints will be presented. Moreover, since there is no mode truncation in the TFM, complete dynamical behavior of the piezoelectric energy harvester can be modeled and analyzed. While the FTGF technique proposed in Ref. [46] can also capture the same dynamics as the TFM, the modal analysis and the FTGF approaches require either obtaining the solution of the eigenvalue problem or of the boundary value problem for the Green's function, respectively, which may involve very tedious procedures, as part of their solution. This implies that changes in the types of boundary conditions of the mechanical model require a re-evaluation of these intermediate solutions, which often involve very time-consuming procedures especially when the boundary conditions are not simple. On the other hand, the TFM algorithm is very amenable to numerical coding; changes in boundary conditions require merely changes in the definitions of the boundary matrices as defined in Eq. (8) and not the algorithm.

4 Electromechanical Systems With Intermediate Constraints

In this section, the application of the TFM is extended to coupled electromechanical harvester systems with nonclassical boundary conditions and intermediate constraints to demonstrate its usefulness as an effective design tool. The presence of constraints allows simple frequency tuning to maximize energy harvesting. For electromechanical beams with intermediate constraints, both the modal analysis method and the FTGF would need to partition the continuous medium into several subsystems (depending on the number of constraints) and solve the eigenvalue or boundary value problems of the subsystems. Such procedures are mathematically laborious. On the other hand, a closed-loop feedback approach can easily be incorporated in the TFM without partitioning the beam harvester into multiple subsystems. These features render the TFM as a powerful and efficient tool for modeling and design optimization of coupled piezoelectric vibration energy harvesting beam systems.

4.1 General Closed-Loop Feedback Formulation. It is noted from either Eq. (37) or Eq. (54) that the coupling voltage is expressed in terms of the strains of the mechanical beam. This coupling can be viewed as a feedback to the beam response, see either Eq. (34) or Eq. (51). A closed-loop feedback approach can, thus be utilized to treat the electrical coupling as well as any mechanical coupling such as intermediate mass, translational and rotational springs, dampers, and any combination of these elements. Employing this approach, the interaction forces, whether electrical or mechanical, are modeled as linear functions of the beam deflection and its derivatives, depending on the nature of the couplings. The response is a function of the coupling forces, expressed in the following form of a coupled algebraic equation in terms of the generalized displacements (for a beam model, its response and first derivative constitute the group of generalized displacements [55,57])

$$\bar{w}(x, s) = \Psi \left( \bar{w}(x_1, s), \ldots, \bar{w}(x_n, s), \frac{\partial \bar{w}(x_1, s)}{\partial x}, \ldots, \frac{\partial \bar{w}(x_n, s)}{\partial x} \right)$$

(62)

where \(n\) denotes the number of couplings. Using Eq. (62), the generalized displacements at those coupling locations are calculated and then feedbacked into the system to give the solution. To do this, the first derivative of \(\bar{w}(x, s)\) is first taken

$$\frac{\partial \bar{w}(x, s)}{\partial x} = \frac{\partial \Psi \left( \bar{w}(x_1, s), \ldots, \bar{w}(x_n, s), \frac{\partial \bar{w}(x_1, s)}{\partial x}, \ldots, \frac{\partial \bar{w}(x_n, s)}{\partial x} \right)}{\partial x}$$

(63)

A system of linear equations is then constructed by substituting the values of \(x_1, \ldots, x_n\) on the left-hand sides of Eqs. (62) and (63). Solving the resulting system of linear equations yields the values of the unknown coupling forces which are then feedbacked into system to determine the response.

4.2 Examples and Results. In Sec. 3, the TFM is successfully applied to and validated for a unimorph cantilever vibration energy harvester. In this subsection, this systematic methodology is further applied to problems with nonclassical boundary conditions and intermediate constraints.

4.2.1 Example 1: Cantilever Beam With a Tip Mass. To increase the harvested vibration energy, one common practice found in the literature is to add a concentrated mass at the tip of the cantilever to increase the amplitude of the vibration and hence the harvested electrical charge. Consider the system shown in Fig. 4
Note that, comparing to the no tip-mass example, the only boundary condition matrices (M(s) and N(s) in Eq. (22)) and the analysis and algorithm remain unchanged. The boundary conditions of the beam with a tip mass are given by

\[
\begin{align*}
  w(x, t)_{x=0} &= 0 \\
  \frac{\partial w(x, t)}{\partial x}_{x=0} &= 0
\end{align*}
\]

\[
\begin{align*}
  &\left[ EI \frac{\partial^2 w(x, t)}{\partial x^2} + c_t J \frac{\partial^2 w(x, t)}{\partial x \partial t} + J_t \frac{\partial^3 w(x, t)}{\partial x \partial t^2} \right]_{x=0} = 0 \\
  &\left[ EI \frac{\partial^3 w(x, t)}{\partial x^3} + c_t J \frac{\partial^3 w(x, t)}{\partial x^2 \partial t} - M_s \frac{\partial^2 w(x, t)}{\partial t^2} \right]_{x=L} = 0
\end{align*}
\]

Taking the Laplace transform of Eq. (66) leads to the following boundary matrices:

\[
M(s) = \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}, \quad N(s) = \begin{bmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}
\]

(65)

Taking the Laplace transform of the aforementioned Eq. (66) gives

\[
\left[ K_T w(x, t) + \frac{\partial}{\partial x} \left( EI(x) \frac{\partial^2 w(x, t)}{\partial x^2} \right) \right]_{x=0} = 0
\]

\[
\left[ K_R \frac{\partial w(x, t)}{\partial x} - EI(x) \frac{\partial^2 w(x, t)}{\partial x^2} \right]_{x=L} = 0
\]

\[
\frac{\partial^2 w(x, t)}{\partial x^2} \mid_{x=L} = 0
\]

\[
\frac{\partial^3 w(x, t)}{\partial x^3} \mid_{x=L} = 0
\]

(66)

For an ideal cantilever, the coefficients of the translational and rotational springs are assumed to be very large. As the values of $K_T$ and $K_R$ in Eq. (67) approaches infinity, the temporal-spatial boundary operator $M(s)$ becomes asymptotically the same as that defined in Eq. (22)

\[
\begin{align*}
  \lim_{K_T, K_R \to \infty} M(s) &= \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & \frac{EI}{K_T} & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix} \\
  \lim_{K_T, K_R \to \infty} N(s) &= \begin{bmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}
\end{align*}
\]

(67)
4.2.3 Example 3: Frequency Tuning Via Constraint of an Intermediate Rotational Spring. The transfer function method is a powerful tool in assessing the effects of frequency tuning in energy harvester designs. Consider the system with a rotational spring at $x = x_r$, see Fig. 6.

By considering the base excitations as boundary conditions (refer to formulations in Sec. 3.3.2), the equation of the piezoelectric beam is

$$m \frac{\partial^2 W(x, t)}{\partial t^2} + c_4 \frac{\partial W(x, t)}{\partial t} + c_4 d \frac{\partial^4 W(x, t)}{\partial x^4} + E I \frac{\partial^4 W(x, t)}{\partial x^4} = \psi(t) \left[ \frac{d \delta(x - x_2)}{dx} - \frac{d \delta(x - x_1)}{dx} \right] + K_r \frac{\partial W(x, t)}{\partial x} \times \frac{d \delta(x - x_r)}{dx}$$

(68)

The boundary conditions are

$$W(x, t) \bigg|_{x=0} = \psi(t), \quad \frac{\partial W(x, t)}{\partial x} \bigg|_{x=0} = \theta(t), \quad \frac{\partial^2 W(x, t)}{\partial x^2} \bigg|_{x=L} = 0, \quad \frac{\partial^3 W(x, t)}{\partial x^3} \bigg|_{x=L} = 0$$

(69)

Upon taking the Laplace transform, the steady-state response of the beam can be written as

$$\tilde{W}(x, s) = \frac{1}{c_4 s + EI} \int_0^L g_{14}(x, \xi, s) \left\{ \theta(\xi) \times \left[ \frac{d \delta(\xi - x_2)}{d\xi} - \frac{d \delta(\xi - x_1)}{d\xi} \right] + K_r \frac{\partial \tilde{W}(\xi, s)}{\partial \xi} \times \frac{d \delta(\xi - x_r)}{d\xi} \right\} d\xi$$

$$+ h_{11}(x, s) \tilde{\gamma}(s) + h_{12}(x, s) \tilde{\theta}(s)$$

(70)

Substituting the voltage from Eq. (54) and using the Dirac delta function property, Eq. (70) can be expressed as

$$\tilde{W}(x, s) = \frac{\frac{\partial \tilde{W}(x, s)}{\partial x} \bigg|_{\xi=x_1} - \frac{\partial \tilde{W}(x, s)}{\partial x} \bigg|_{\xi=x_2}}{(c_4 s + EI)(\xi_s + 1)} \times \left[ g_{23}(x, x_2, s) - g_{23}(x, x_1, s) \right] + \frac{K_r}{c_4 s + EI} g_{14}(x, x_r, s) \frac{\partial^2 \tilde{W}(\xi, s)}{\partial \xi^2} \bigg|_{\xi=x_r}$$

$$+ h_{11}(x, s) \tilde{\gamma}(s) + h_{12}(x, s) \tilde{\theta}(s)$$

(71)

The slopes of the beam at $x = x_1, x_2, x_r$ and the second derivative of the beam deflection at $x = x_r$ are unknowns in Eq. (71). The closed-loop feedback approach is now applied to calculate these unknowns and back-substitute them into Eq. (71) to find the closed-form solution for the deflection of the beam. By the state-space frequency domain formulation, the slope and the second derivative of the beam deflection can be expressed as

$$\frac{\partial \tilde{W}(x, s)}{\partial x} = \frac{\frac{\partial \tilde{W}(x, s)}{\partial x} \bigg|_{\xi=x_1} - \frac{\partial \tilde{W}(x, s)}{\partial x} \bigg|_{\xi=x_2}}{(c_4 s + EI)(\xi_s + 1)} \times \left[ g_{23}(x, x_2, s) - g_{23}(x, x_1, s) \right] - \frac{K_r}{c_4 s + EI} g_{24}(x, x_r, s) \frac{\partial^2 \tilde{W}(\xi, s)}{\partial \xi^2} \bigg|_{\xi=x_r}$$

$$+ h_{21}(x, s) \tilde{\gamma}(s) + h_{22}(x, s) \tilde{\theta}(s)$$

(72)

$$\frac{\partial^2 \tilde{W}(x, s)}{\partial x^2} = \frac{\frac{\partial \tilde{W}(x, s)}{\partial x} \bigg|_{\xi=x_1} - \frac{\partial \tilde{W}(x, s)}{\partial x} \bigg|_{\xi=x_2}}{(c_4 s + EI)(\xi_s + 1)} \times \left[ g_{33}(x, x_2, s) - g_{33}(x, x_1, s) \right] - \frac{K_r}{c_4 s + EI} g_{34}(x, x_r, s) \frac{\partial^2 \tilde{W}(\xi, s)}{\partial \xi^2} \bigg|_{\xi=x_r}$$

$$+ h_{31}(x, s) \tilde{\gamma}(s) + h_{32}(x, s) \tilde{\theta}(s)$$

(73)

Substituting $x = x_1, x = x_2$, and $x = x_r$ into Eq. (72) and $x = x_r$ into Eq. (73) results in a system of four linear equations and four unknowns. Solving for these unknowns by standard techniques in linear algebra and back-substituting the results into Eqs. (71)–(73), one can determine the deflection, slope, and second derivative at any given point. The harmonic steady-state response for the relative displacement of the beam is then obtained by substituting the result for $\tilde{W}(x, s)$ into Eq. (52). This model can be validated through known results. For instance, if one places the rotational spring at the free end of the beam $x_r = L$ and assume that the rotational stiffness has a very large value, one would expect to get the same response as the response of a clamped-sliding piezoelectric beam. The clamped-sliding beam response can be obtained by modifying the cantilevered unimorph boundary conditions as follows:

$$M(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad N(s) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For demonstration of results, consider the specific case of $x_1 = 0$ and $x_2 = L$. Figure 7 plots the response of a piezoelectric cantilevered beam with the rotational spring at $x_r = L$. The result matches with the response of a clamped-sliding beam as the rotational stiffness becomes sufficiently large. Another benchmark result is the response of a piezoelectric cantilevered beam which was presented in Fig. 3. In our computer codes, only simple changes to set the location of the rotational spring at $x_r = L$ and $K_r = 0$ are needed. The validation is shown in Fig. 8.

Figure 9 shows shifts in the response peaks from the cantilevered piezoelectric beam model to the clamped-sliding cantilevered beam when the rotational stiffness changes from zero to a very large value. The location of the rotational spring is at $x_r = L$. This figure
represents an important original result showing the ease of using the proposed methodology in energy harvester designs via frequency tuning. By properly selecting and defining the parameters of the harvesters in the computer codes, engineers and researchers can use the TFM as an effective tool for analyses and designs. The voltage FRF is shown in Fig. 10. In both plots, the resonance frequencies shift up as the stiffness of the rotational spring increases. Note that the output voltage decreases as $K_r$ increases.

The amplitude of the output voltage for the clamped-sliding beam is zero, which cannot be seen in Fig. 10 due to the logarithmic scale. This result can be proven analytically. From Eq. (54), the relationship between the voltage and the beam total deflection is

$$v_s(x) = \frac{u_{cs}}{s} \left( \frac{\partial W_s}{\partial x} \right)_{x_1} - \left. \frac{\partial W_s}{\partial x} \right|_{x_2}$$

Since the clamped-sliding beam has zero slope at both of its boundaries, $(\partial W_s(x,s)/\partial x)|_{x_1=0} = 0$ and $(\partial W_s(x,s)/\partial x)|_{x_2=L} = 0$, the output voltage is equal to zero.

5 Conclusions

In this paper, an exact, closed-form, semi-analytical solution of the frequency response function of vibration energy harvesting beam systems is presented using the distributed transfer function method. The solution methodology is applicable for coupled electromechanical systems with nonproportional damping, intermediate constraints, and nonclassical boundary conditions. This technique is applied to formulate the problem of piezoelectric energy harvesting beams under harmonic base excitations. It is shown that the problem can be formulated by either treating the excitations as external forces in the field equations or as nonhomogenous boundary conditions. In general, the latter approach leads to simpler formulations and evaluations of the solutions. The proposed method has numerous advantages over other methods available in the literature. The solution is exact, with no truncation or approximation needed, and requires no knowledge of the system eigensolutions which may be difficult to obtain for beams with nonclassical boundary conditions. Moreover, frequency response and higher-order derivatives such as strains of the beam systems can be obtained simultaneously, without performing any differentiation. Our method is systematic, rendering it amenable to algorithmic numerical coding. Different boundary conditions require only minor modifications in the definitions of the parameters in the boundary operators, and there is no need to...
derive the characteristic equation in the evaluation of the coupled system transfer function. For electromechanical energy harvesting beams with intermediate constraints, it is shown that a closed-loop feedback approach can easily be incorporated in the transfer function method without partitioning the beam into multiple subsystems. These features render the method as a powerful tool for modeling and optimized design of vibration energy harvesting beam systems.

Numerical results of the frequency response of piezoelectric energy harvesting beams under end excitations are presented and match very well with published results. Moreover, our results reveal detailed information in the frequency response which could only be obtained when higher-order approximations are employed in the modal analysis method. The presence of intermediate constraints allows simple frequency tuning to maximize energy harvesting. Examples are shown to demonstrate the use of the proposed method as a design tool for vibration energy harvesters via tuning of system parameters. Since our results are exact, they can also be used for future reference to provide benchmark data for assessing the accuracies of approximate techniques.

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Nomenclature

Beam Harvester Parameters in Standard SI Units

- \( b \) = width of the beam harvester (m)
- \( c_w \) = viscous air damping coefficient
- \( c_i \) = equivalent strain-rate damping coefficient
- \( d_{33} \) = transverse piezoelectric constant (m/V)
- \( E \) = Young’s modulus (N/m²)
- \( E_p \) = Young’s modulus of the piezoelectric layer (N/m² or Pa)
- \( E_s \) = Young’s modulus of the substructure layer (N/m² or Pa)
- \( f(x,t) \) = transverse external force per unit length (N/m)
- \( h_p \) = thickness of the piezoelectric layer (m)
- \( h_{ps} \) = distance from the neutral axis to the centerline of the piezoelectric layer (m)
- \( h_s \) = thickness of the substrate layer (m)
- \( h_{s2} \) = location of translational spring (m)
- \( I \) = cross section moment of inertia (m⁴)
- \( I_m \) = moment of inertia of proof mass (kg/m²)
- \( K_r \) = rotational spring constant of an intermediate constraint (N·m/rad)
- \( K_{r_b} \) = rotational spring constant at the boundary (N·m/rad)
- \( K_f \) = translational spring constant at the boundary (N/m)
- \( L \) = length of the beam harvester (m)
- \( m \) = mass per unit length (kg/m)
- \( M_f \) = mass of proof mass (kg)
- \( R_e \) = electric resistance (Ω)
- \( u_0(x) \) = initial displacement of the distributed parameter system (m)
- \( v_0(x) \) = initial velocity of the distributed parameter system (m/s)
- \( w(x,t) \) = bending deflection of the piezoelectric beam relative to base excitation (m)
- \( w_3(x,t) \) = base excitation of the piezoelectric beam (m)
- \( W(x,t) \) = total bending deflection of the piezoelectric beam (m)

System Transfer Functions

- \( A \) = spatial differential operator
- \( B \) = spatial differential operator
- \( C \) = spatial differential operator
- \( F(x) \) = state coefficient matrix
- \( G(x,\xi,s) \) = system transfer function (closed-loop transfer function)
- \( H(x,s) \) = boundary transfer function for the response and its derivatives
- \( M_f \) = temporal-spatial linear differential operators for boundary conditions at \( x = 0 \)
- \( N_f \) = temporal-spatial linear differential operators for boundary conditions at \( x = L \)
- \( q(x,s) \) = vector of external forces
- \( W_o(x,\xi,s) \) = transfer function of the distributed parameter structural system
- \( Y_{R_f}(t) \) = temporal-spatial linear differential operators for boundary inhomogeneity
- \( \Phi(x,s) \) = fundamental matrix
- \( \phi(x,s) \) = boundary transfer function for the system response
- \( \eta(x,s) \) = state vector for beam deflection and its derivatives

References


