Geometrically nonlinear mechanical properties of auxetic double-V microstructures with negative Poisson’s ratio

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ARTICLE INFO

Keywords:
Auxetic materials
Negative Poisson’s ratio
Nonlinear geometry modeling
Double-V microstructure
Large strain
Mechanical properties

ABSTRACT

Double-V microstructure (DVM) is a type of auxetic cellular material with negative Poisson’s ratio (NPR). Accurate predictions of the mechanical properties of these microstructures at large strains are critical for many engineering applications. In this paper, nonlinear theoretical models of two-dimensional (2D) and three-dimensional (3D) DVM based on a large beam deflection model are established to predict the normalized Young’s modulus and Poisson’s ratio in the principal directions. The theoretical solutions are compared to solutions obtained from numerical finite element analyses and quasi-static compression experiments of a 2D prototype manufactured using additive printing technique. It is found that there is good agreement between these results, validating the accuracy of the proposed theoretical model. Effects of geometry parameters on the mechanical properties of the DVM are also investigated to understand the mechanical behavior at large strains. This study provides validated models for predicting the behavior of these microstructures at large strains, useful for engineering designs.

1. Introduction

In recent years, research in auxetic cellular microstructures has attracted increased attention due to their unconventional mechanical properties. Unlike traditional structures and materials, auxetic microstructures possess the property of negative Poisson’s ratio such that the materials shrink in the perpendicular direction when compressed and expand when stretched. Moreover, auxetic structures and materials have been shown to have higher shear stiffness (Lakes, 1987; Scarpa and Tomlin, 2000) and fracture toughness (Bianchi et al., 2008; Levy et al., 2006), indentation resistance (Hu and Deng, 2015; Photiou et al., 2016), unique energy absorption ability (Bezazi and Scarpa, 2009, 2007; Scarpa et al., 2005, 2004), and good acoustic absorption abilities (Howell et al., 1994; Scarpa and Smith, 2004). These improved structural properties of auxetic microstructures have made them appealing for a wide range of engineering applications. In 1987, an isotropic foam model with negative Poisson’s ratio (NPR) was proposed for the first time (Lakes, 1987). Since then, mechanical properties of auxetic structures and materials with various configurations have been investigated by theoretical, numerical, and experimental methods.

Among these auxetic structures and materials with different cell types, the re-entrant hexagonal and double-V microstructures have been proven to possess higher Young’s moduli (Elipe and Lantada, 2012). A theoretical model to predict the Young’s modulus and Poisson’s ratio of conventional re-entrant hexagonal microstructures was developed based on the bending theory of Euler-Bernoulli beams (Gibson and Ashby, 1999). To improve the accuracy of their predictions, a refined bending model considering the effects of the stretching and the relative rotational motions at the hinges of the beams was developed (Masters and Evans, 1996). In addition, buckling and shear strengths of re-entrant microstructures were also investigated. A model to predict the buckling strength of cellular structures under a general macroscopic in-plane stress state was developed based on the classical beam theory (Haghpanah et al., 2014). Transverse elastic shear moduli of auxetic re-entrant microstructures were theoretically predicted based on the Voigt and Reuss bounds of stiffness. The analytical predictions were validated by a full-scale finite element technique (Lira et al., 2009). The in-plane flexible shear stress of the hexagonal microstructure was investigated based on the homogenization method (Ju and Summers, 2011). It was found that re-entrant hexagonal microstructures with NPR have higher macroscopic shear flexible properties associated with higher vertical cell walls.

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https://doi.org/10.1016/j.euromechsol.2019.103933
Received 18 March 2019; Accepted 15 December 2019
Available online 18 December 2019
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Much research has been conducted to enhance the mechanical properties of re-entrant hexagonal microstructures. Particularly, the gradient configuration was introduced in the microstructure to improve the in-plane stiffness, bending resistance and fracture toughness. The thickness gradient re-entrant microstructure was proposed to enhance the transverse shear moduli (Lira and Scarpa, 2010). Finite element models and experimental prototypes of gradient re-entrant microstructures with linearly increasing wall thickness along the panels were constructed to validate the conclusion. In order to improve the in-plane stiffness of the normal re-entrant hexagonal honeycomb (NRHH), Lu (Lu et al., 2016) and Zied (Zied et al., 2015) proposed novel structures which they named splined-reentrant and stiffened-reentrant microstructures based on the basic NRHH. The Young’s modulus and Poisson’s ratio of these structures were compared favorably with results obtained by analytical and numerical techniques. Inspired by this, the in-plane stiffness and buckling strength was enhanced by embedding the rhombic configuration into the basic NRHH (Fu et al., 2017). It was found that structures with this configuration can have higher normalized Young’s modulus compared to the novel structures proposed in the studies (Lu et al., 2016; Zied et al., 2015).

Another promising auxetic structure is the double-V microstructure (DVM). Compared to NRHH, the deformation of DVM during a crushing procedure is more stable. Due to the unique properties of DVM microstructures, they have been applied in various applications in automotive engineering (Wang et al., 2018, 2017; Zhou et al., 2017, 2016). Its advantages for the application to a blast-protective deflector were discussed (Ma et al., 2009). As for the mechanical properties of the DVM, Young’s modulus, Poisson’s ratio, and yield stress of the 2D DVM were predicted analytically (Qiao and Chen, 2015a, b). However, the interaction effects of adjacent unit cells on the mechanical properties were neglected. This may contribute to the underestimation of the Young’s modulus. The theoretical model was revised and extended it to 3D DVM (Gao et al., 2018). The analytical model, validated by numerical and experimental results, was shown to predict the Young’s Modulus and Poisson’s ratio more accurately. A set of suitable homogenization methods which provides an effective means for the multiscale modeling of DVM microstructures was developed (Ma, 2017). The effects of geometry parameters on the Poisson’s ratio of 3D DVM were investigated, however, without considering the mechanics of beams (Lim, 2016).

The above cited research, using either analytical modeling and prediction with numerical methods, focuses on the elastic regime of the mechanical properties of auxetic materials. However, in many applications, the DVM microstructures are employed in buildings and infrastructures where they could experience large deformations. Theoretical models based on elastic and impact analyses are thus insufficient to provide good predictions. Therefore, development of theoretical models to predict the mechanical properties of DVM microstructures under large deflections is desirable. For re-entrant hexagonal microstructures, Wan (Wan et al., 2004) and Levy (Levy et al., 2006) established a multi-directional model based on a large deflection beam theory to predict the mechanical properties. However, in these models, the predicted results were physically impossible because the maximum deflection angles of the structures were treated as input parameters which are defined rather arbitrarily instead of being determined by the geometry parameters. An improved analytical solution based on both a large deflection beam model and the Timoshenko beam model was proposed to predict the Young’s modulus of 3D re-entrant microstructures (Yang et al., 2015). Their results were validated by experimental and finite element analyses. To date, analytical models to predict the mechanical properties of 2D and 3D DVM under large deformations have not been established. DVM microstructures have been employed in applications where large deformations are experienced, and it is thus essential to develop models that can predict accurately the mechanical behavior of these materials under these circumstances. Moreover, designs of DVM microstructures as smart structures require such large-deformation analytical models to accurately predict and control the mechanical properties under dynamic situations.

In this paper, an analytical model based on a large deflection beam model is established to predict the mechanical properties in all principal directions of DVMs in the geometrically nonlinear regime. Full-scale finite element models of the DVMs are constructed to validate the analytical predictions of the Young’s modulus and the Poisson’s ratio in all the principal directions. Moreover, effects of the geometry parameters on these mechanical properties are investigated. A quasi-static experiment of a 2D DVM prototype is conducted to validate and compare with the predictions obtained from the proposed analytical model.

2. Geometric configurations of auxetic double-V microstructure

The two-dimensional (2D) auxetic double-V microstructure (DVM) shown in Fig. 1 was first proposed in 1996 (Larsen et al., 1996). The 2D DVM is arranged by the double-V unit cell in X and Y directions periodically. The short and long inclined beams are denoted by S- and L-beams, respectively. As shown in Fig. 1, the unit cell is defined by three geometry parameters: l, \( \theta_s \), \( \theta_l \), where l is the horizontal distance between the two vertices A and D, \( \theta_s \) is the respective angle between the S-beam and vertical axis AC, \( \theta_l \) is the respective angle between the L-beam and vertical axis AC. The 2D DVM can be extended to the three-dimensional (3D) DVM by intersecting the unit cells, as shown in Fig. 2. To simplify the problem, the cross-sections of all the struts are assumed to be square, represented by \( t \times t \). It can be understood that 2D DVM exhibits the negative Poisson’s ratio property in both the two principal directions. Consider the Y direction as an example, when the
vertex A is compressed, the vertices B and D tend to get close to the axis AC. This explains why DVM microstructures have negative Poisson’s ratio property. Similarly, for 3D DVM, in all the principal directions, the structures have negative Poisson’s ratio property. In particular, the mechanical properties in both the X and Y directions are identical due to symmetry.

The following assumptions are imposed in our modeling to investigate the mechanical properties of DVMs theoretically and numerically. The double-V unit cells are considered to be located in an infinite macro DVMs, allowing us to eliminate boundary effects and retain maximum structural symmetry. All of the joints in the DVM are assumed to be rigid, and the deformation of the struts is primarily contributed by bending. The effects of axial and shear are small compared to bending and are thus ignored.

3. A large deflection cantilever beam model

The mechanical properties (Young’s modulus and Poisson’s ratio) of DVM microstructures in the elastic regime were predicted by an analytical model which treated each beam of the DVM microstructure as a cantilever Euler-Bernoulli beam (Gao et al., 2018). However, an analytical model to predict the mechanical properties of the DVM microstructure under large deflections needs to treat the beams using large deflection modeling. A large deflection cantilever beam model is discussed as follows.

Fig. 3 shows the kinematics and free-body diagram of a cantilever beam under a combined force and moment loads (with the positive conventions shown).

![Fig. 3. Large deflection of a cantilever beam under combined force and moment loads (with the positive conventions shown).](image)

Here, M is the moment, \( \frac{d\theta}{ds} \) is the rate of change of angular deflection (slope) along the beam length, \( EI \) is the flexural rigidity of the beam, y is the transverse deflection, and x is the coordinate along the undeflected beam axis. The difference between the small and large deflections of the beam is the term \( \frac{dy}{dx} \). For small deflections, \( \left( \frac{dy}{dx} \right)^2 \) is small enough to be neglected which leads to the standard linearized beam theory. However, in large deflections, this term cannot be neglected. As shown in Fig. 3, the bending moment at an arbitrary point A (x, y) is given by:

\[
M = EI \frac{d\theta}{ds} = P(a-x) + nP(b-y) + M_0 \quad (2)
\]

where \( n \) is the ratio of the horizontal force over the vertical force. Eq. (2) can be rewritten as

\[
\frac{d^2\theta}{ds^2} = \frac{P}{EI} \left( \frac{dx}{ds} \frac{dy}{ds} n \right) \quad (3)
\]

Note that

\[
\frac{d\theta}{ds} = \frac{dx}{ds} \cos\theta \quad (4)
\]

\[
\frac{d\theta}{ds} = \frac{dy}{ds} \sin\theta \quad (5)
\]

Incorporating Eqs. (3)-(5) into Eqs. (1) and (2), the following results can be obtained:

\[
\frac{1}{2} \left( \frac{d\theta}{ds} \right)^2 = \frac{P}{EI} \left( -\sin\theta + n \cos\theta + \sin\theta_0 - n \cos\theta_0 + \frac{M_0}{EI} \right) \quad (6)
\]

where \( \theta_0 \) is the deflection angle of the beam at its free end. Eq. (6) can be rewritten as

\[
\frac{d\theta}{ds} = \pm \sqrt{\frac{P(-\sin\theta + n \cos\theta + \omega)}{EI}} \quad (7)
\]

where

\[
\omega = \sin\theta_0 - n \cos\theta_0 + k \quad (8)
\]

and \( k \) is the load ratio (“Compliant Mechanisms - Larry L. Howell - Google Books,” n.d.) given by

\[
k = \frac{M_0^2}{2PEI} \quad (9)
\]

The sign in Eq. (7) is defined as in (Byrd and Friedman, 1954);
negative for convex downward curvature and positive for concave upward curvature. Also, the force index $\alpha$ is defined as

$$\alpha = \frac{pEJ}{EI}$$  \hspace{1cm} (10)

Combining Eqs. (4), (5) and (7), the general equations for large deflections of the cantilever beam can be obtained

$$\alpha = \pm \frac{1}{\sqrt{2}} \int_0^{\chi_0} \frac{d\theta}{\sqrt{-\sin\theta + \cos\theta + \omega}}$$  \hspace{1cm} (11)

$$a = \pm \frac{1}{\sqrt{2}} \int_0^{\chi_0} \frac{\cos\theta d\theta}{\sqrt{-\sin\theta + \cos\theta + \omega}}$$  \hspace{1cm} (12)

$$b = \pm \frac{1}{\sqrt{2}} \int_0^{\chi_0} \frac{\sin\theta d\theta}{\sqrt{-\sin\theta + \cos\theta + \omega}}$$  \hspace{1cm} (13)

where $a/L$ and $b/L$ are the nondimensional coordinates of the tip point along the $x$ and $y$ directions. The sign of Eqs. 11–13 are determined in the same way as for Eq. (7). To solve these equations, the elliptical integrals are introduced. The incomplete elliptic integrals of the first and second kinds are defined as (Byrd and Friedman, 1954):

$$F(\gamma, \chi) = \int_0^{\gamma} \frac{d\delta}{\sqrt{1 - \chi^2 \sin^2 \delta}}$$  \hspace{1cm} (14)

$$E(\gamma, \chi) = \int_0^{\gamma} \frac{1 - \chi^2 \sin^2 \delta d\delta}{\sqrt{1 - \chi^2 \sin^2 \delta}}$$  \hspace{1cm} (15)

respectively, where $\gamma$ is called the amplitude and $\chi$ ($-1 \leq \chi \leq 1$) the modulus. When $\gamma = \pi/2$, they can be denoted as $F(\chi)$ and $E(\chi)$ which represent the complete elliptic integrals of the first and second kinds, respectively. In addition, according to the elastica theory, the solutions of the deflections of the beams are also related to the inflection points. In the special case of the DVM, the deflection angles at all ends of the beams are zero because of the geometrical symmetry. Therefore, the number of inflection points of the beams in DVM is 1. The analytical solution of the deflections for the cantilever beams with one inflection point can be summarized as:

$$\alpha = -\frac{S_M}{\sqrt{f}}$$  \hspace{1cm} (16)

$$a = \frac{-S_M L}{a^2 f} \left(-n^2 f + 2e_{\gamma} + 2\sqrt{\omega_e}\right)$$  \hspace{1cm} (17)

$$b = \frac{-S_M L}{a^2 f} \left(-n^2 f - 2e_{\gamma} + 2\sqrt{\omega_e}\right)$$  \hspace{1cm} (18)

where

$$f = F(\gamma_2, \gamma) - F(\gamma_1, \chi) - 2S_M F(\chi)$$  \hspace{1cm} (19)

$$e = -E(\gamma_2, \chi) - E(\gamma_1, \chi) - 2S_M E(\chi)$$  \hspace{1cm} (20)

$$\eta = \sqrt{1 + n^2}$$  \hspace{1cm} (21)

$$S_M = \begin{cases} 1 & M_0 \geq 0 \\ -1 & M_0 \leq 0 \end{cases}$$  \hspace{1cm} (22)

$$\chi = \sqrt{\frac{\omega^2 + n}{2\eta}}$$  \hspace{1cm} (23)

$$c = \sqrt{\omega + n + \sqrt{\omega - \sin \theta_0 + \cos \theta_0}}$$  \hspace{1cm} (24)

$$r_1 = \sin^{-1} \left(\frac{\eta - n}{\omega + n}\right)$$  \hspace{1cm} (25)

$$r_2 = \sin^{-1} \left(\frac{\eta + \sin \theta_0 - \cos \theta_0}{\omega + n}\right)$$  \hspace{1cm} (26)

4. Nonlinear modeling of 2D and 3D double-V microstructures

4.1. Mechanical properties in the $z$-direction

The solution procedure to determine the mechanical properties in the vertical loading direction based on the large deflection cantilever beam model is summarized in this section. As shown in Fig. 1, there is a compressive stress $\sigma_w$ applied to the 2D DVM in the $z$-direction. Invoking the periodic symmetry of the DVM, only half of the unit cell with the corresponding loading and boundary conditions needs to be considered for the macro DVM. Fig. 4(a) depicts the loading and boundary conditions of the half double-V unit cell: A (1,0,1) indicates that the vertex $A$ is fixed in X-direction and cannot rotate, B (0,0,1) only the deformation rotation is confined at the vertex $B$, and C (1,1,1) the
Forces and moments, that are to be determined, in the free-body diagrams (Gao et al., 2018).

From the force equilibrium, the force $P_A$ can be readily obtained as:

$$P_A = \sigma_w l$$

Based on the large deflection model for a cantilever beam, the beams $AB$ and $BC$ are treated as cantilever beams to predict the mechanical properties, Young’s modulus and Poisson’s ratio, of DVM under large deflections. According to the geometric relationships and force equilibrium, the vertical force $P$, horizontal force $nP$ and end moment $M_0$ of beam $AB$ can be obtained as follows:

$$P_1 = P_A \sin \theta_1 + F_A \cos \theta_1$$

$$n_1P_1 = -P_A \cos \theta_1 + F_A \sin \theta_1$$

$$M_{01} = M_A$$

Referring to Fig. 3 on the positive convention of $P$, $nP$ and $M_0$ and using Eqs. 16–18, the deflection displacements $a_1$, $b_1$ and rotation angle $\beta_1$ in local coordinate of the cantilever beam $AB$ can be calculated. The force and moment at the end of the beam $BC$ are obtained as follows:

$$P_2 = P_A \sin \theta_2 + F_A \cos \theta_2$$

$$n_2P_2 = -P_A \cos \theta_2 + F_A \sin \theta_2$$

$$M_{02} = -M_A - P_1a_1 - n_1P_1b_1 + M_B$$

After calculating the deflection displacements $a_2$, $b_2$ and rotation angle $\beta_2$ of beam $BC$, three equations yielding the constraints of the half double-V unit cell can be obtained. The horizontal displacement of the vertex $A$ and rotation angles of vertices $A$ and $B$ in the global coordinate system in Fig. 4 are zero. In summary, the three unknown parameters $F_A$, $M_A$ and $M_0$ can be obtained from the three Eqs. 34–36 listed below. The variables $\beta_1$, $\beta_2$, $a_2$, $b_2$ and $\theta_0$ in Fig. 3 are different from the condition in the last section. According to Eq. 7, they are readily obtained:

$$\beta_1 = 0$$

$$\beta_1 + \beta_2 = 0$$

The vertical displacement $\Delta z_1$ of vertex $A$ and the horizontal displacement $\Delta x_1$ of vertex $B$ for the DVM can be written as:

$$\Delta x_1 = -(l_2 - a_2) \sin \theta_1 + b_1 \cos \theta_1 + (l_2 - a_2) \cos \theta_2 + b_2 \sin \theta_2$$

$$\Delta z_1 = -(l_2 - a_2) \sin \theta_2 + b_2 \cos \theta_2$$

Define the normalized Young’s modulus as the ratio of the Young’s modulus of the DVM microstructure over that of the base material. The normalized Young’s modulus and Poisson’s ratio can readily be obtained from the following equations.

$$\frac{E_T}{E} = \frac{\Delta x_1}{\Delta z_1}$$

where $E$ is Young’s modulus of the base material for the DVM.

### 4.2. Mechanical properties in the x-direction

Consider the DVM compressed in x-direction as shown in Fig. 5. The boundary condition is the same as Fig. 4. The only difference is that the external force will be applied in the horizontal direction at the vertex $B$. The three unknown force and moment parameters $F_A$, $M_A$ and $M_0$ can be solved by the three equations governing the placement and rotation angles at vertices $A$ and $B$ by treating the half double-V unit cell as two cantilever beams. Then the normalized Young’s modulus and Poisson’s ratio can be predicted from the force equilibrium, the force $P_B$ can be readily obtained as:

$$P_B = \sigma_w l \frac{\sin(\theta_1 - \theta_2)}{\sin \theta_1 \sin \theta_2}$$

The force and moment at the ends of the beams $AB$ and $BC$ are different from the condition in the last section. According to Fig. 5, they are readily obtained:

$$P_1 = F_A \cos \theta_1$$

$$n_1P_1 = F_A \sin \theta_1$$

$$M_{01} = M_A$$

(44)
The normalized Young’s modulus and Poisson’s ratio in x-direction can then be obtained from the following equations.

\[ E_x = \frac{\sigma_{xx}}{\Delta x/2} 
\]

\[ \nu_{xy} = \frac{\Delta y}{\Delta x} 
\]

### 4.3. Nonlinear modeling of 3D double-V microstructures

The 2D DVM can be extended to 3D DVM by intersecting the unit cells. Compared to 2D DVM, it has a lower relative density and can shrink in two directions when compressed. In engineering applications, 3D DVM exhibits better energy absorption compared to 2D DVM. In order to design DVMs more effectively, it is essential to study the mechanical properties of 3D DVMs under large deflections. Due to the symmetry in the geometry, the Young’s modulus and the Poisson’s ratio in both the x and y directions are identical. Based on the large deflection model for the cantilever beams, the only difference of the predictions for the 2D DVM and 3D DVM lies in the equations relating the external concentrated force \( P_A \), \( P_B \) and stress \( \sigma_{xx} \). In the nonlinear analytical model for the 3D DVM, Eqs. (27) and (41) are to be replaced by the following equations.

\[ P_A = \sigma_{xx} A^2 \sin \theta_2 \]

\[ \sigma_{xx} = \frac{\sin \theta_2}{2 \sin \theta_1 \sin \theta_2} \]

Therefore, the normalized Young’s modulus and Poisson’s ratio of 3D DVM can be obtained based on the calculation procedures outlined for the 2D DH earlier.

### 5. Results and discussion

#### 5.1. Finite element model

Numerical finite element models are constructed to predict and verify the analytical predictions of the mechanical properties of the 2D/3D DVM based on the large deflection beam model for the DVM. The finite element model for the 2D DVM has 12 units in each direction and thus a total of 144 unit cells, as illustrated in Fig. 6(a). The ABAQUS software is employed to analyze the mechanical properties under compression. Because the S- and L-beams are sufficiently slender, they are simulated as Euler-Bernoulli beams B23 in the finite element (FE) model, which can characterize the deformation modes of the beams. To guarantee accuracy and efficiency of the simulations, the basic element size is chosen as 1 mm through the meshing convergence test. Boundary conditions are illustrated in Fig. 6, with the nodes in the bottom line constrained in the z-direction. Either the left or right side of the finite element domain is constrained in x-direction and the loading stress is applied to the top of 2D DVM. Properties of the base material for the S- and L-beams are: Young’s modulus \( E_y = 2.2 \) GPa, Poisson’s ratio \( \nu_{y} = 0.39 \), mass density \( \rho_1 = 1.08 \) g/cm\(^3\). From the FE model in Fig. 6, the relations \( \Delta z = L_z - L_x \), \( \Delta x = L_x - L_x \) can be used to calculate the normalized Young’s modulus and Poisson’s ratio of the 2D DVM. The FE model for 3D DVM is similar to the one for 2D DVM; therefore, the details are not discussed herein.

#### 5.2. Comparison of the analytical and numerical solutions

For the purpose of comparing the analytical and finite element numerical results, a model of the 2D DVM of which the unit cell with \( \theta_1 = 60^\circ \), \( \theta_2 = 30^\circ \) and \( l = 30 \) mm is chosen as the base model. A parametric study was conducted to understand the effects of the geometry parameters on the mechanical properties of the DVMs under large deflections. For the plots shown in Fig. 9, the angle of the L-beam \( \theta_2 \) and half width of the unit cell \( l \) are also set to be equal to 30\(^\circ\) and 30\(\)mm, respectively. Comparison of results for the normalized Young’s modulus and Poisson’s ratio for the 2D DVM in the two principal z and x direction is discussed as follows.

5.2.1. Solutions in the z-direction

The deformation pattern of the unit cell compressed in the z-direction is illustrated in Fig. 7. Six deformation states are shown, and the black curve represents the undeformed state. The stress increment between successive states of deformation is 12.5 MPa. Denote the first state as the case when the structure is subjected to 12.5 MPa compression, and the sixth state is thus under 75 MPa. It is evident that the unit cell shrinks in the x-direction when compressed in the z-direction. In addition, the strain in the x-direction decreases gradually until the fifth state. This is explained by the geometry nonlinearity which enhances the stiffness of the structure. The strain in the sixth state increases suddenly due to...
buckling of the beams.

Figs. 9–11 illustrate the relationships between the normalized Young’s modulus and Poisson’s ratio and strain for 2D DVM with different geometry configurations specified by $\theta_1$, $\theta_2$, and $l$. In these figures, the solid lines represent the analytical prediction utilizing the nonlinear theoretical model. Numerical predictions of the mechanical properties with strain equals to 0.01, 0.05, 0.1, 0.12, 0.15, 0.18 and 0.22 have been chosen to validate the accuracy of the analytical predictions, and they are denoted as scatter points. There is very good agreement between the analytical and numerical solutions, verifying the accuracy and validity of the analytical predictions employing the nonlinear geometry model. Also, it is shown that, regardless of the value of the strain, the 2D DVM with larger $\theta_1$ has lower normalized Young’s modulus and higher Poisson’s ratio. With the increase of strain, the normalized Young’s modulus will increase first and maximizes at a specific strain value. This is because the DVM shrinks in $x$-direction due to the negative Poisson’s ratio which enhances the stiffness. In the DVM, the S-beam is stretched, and L-beam is compressed. Therefore, further increase of the strain will cause buckling of the L-beam, causing the stiffness of the macro DVM to decrease. In other words, the normalized Young’s modulus will become smaller.

To quantify comparisons between the solutions of the linear and nonlinear theoretical models, a relative error indicator $\lambda$ is introduced

$$\lambda = \frac{E_{\text{nonlinear}} - E_{\text{linear}}}{E_{\text{linear}}} \times 100\%$$

Compared to the predictions based on the linear model (its value is shown by a dashed horizontal line) in Fig. 9(a), the normalized Young’s modulus $E_z$ of the DVM microstructure with $\theta_1 = 45^\circ$ and $\theta_2 = 75^\circ$ is improved by 92.3% and 73.3%, respectively, when the strain $\varepsilon = 0.15$. This indicates that the nonlinear theoretical model, which takes into

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<th>$\theta_2$ ($^\circ$)</th>
<th>$l$ (mm)</th>
<th>Linear model (1e-3)</th>
<th>Nonlinear model</th>
<th>Theoretical solution (1e-3)</th>
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Fig. 8. Deformation patterns: (a) Comparison of deformation patterns with linear and nonlinear solutions; (b) Comparison of deformation patterns with theoretical and numerical solutions.

Table 1
Comparison of the normalized Young’s modulus of DVM microstructure in $z$ direction ($\varepsilon = 0.1$).
account of the beam geometric nonlinearity to enhance the stiffness of the DVM microstructure, is essential for accurate predictions of the mechanical behavior. It is noticed that the DVM microstructure with smaller $\theta_1$ can increase the normalized Young’s modulus more percentage-wise. These comparisons are summarized in Table 1, where it is shown that the nonlinear theoretical model is accurate enough and negative Poisson’s ratio properties of DVM microstructure can improve the Young’s modulus under large deflections.

The trends observed in the Poisson’s ratio versus strain curves shown in Fig. 9 can be explained by similar physical phenomena discussed above. Moreover, the maximum/minimum values of where the Young’s modulus and Poisson’s ratio occur correspond to the same strain values. For smaller $\theta_1$, these extremum points occur at smaller values of strain. Some mechanical properties of the DVM under larger strains are not shown in the figures because the L- and S- beams have contracted. The deformation modes of the DVM will thus deviate significantly from the nonlinear model, leading to differences between the theoretical and numerical finite element predictions.

Fig. 8(a and b) illustrates the comparisons between the linear and nonlinear deformation patterns obtained by the theoretical and numerical solutions. In Fig. 8(a), pattern A represents the deformation pattern with the large deflection model for the cantilever beam. Pattern B is the one with the elastic theory for the Euler-Bernoulli beam. It is evident that under the same stress, pattern B deforms more. This shows that the NPR structure shrinks when compressed, to enhance the stiffness. Also, in Fig. 8(b), there is good agreement between the deformation pattern predicted by the theoretical solution (red curve) and that obtained by the numerical solution (green curve), where the black curve represents the initial state of the NPR structure. Hence, the large deflection model can accurately predict the deformations and the deformation shapes.

Fig. 10(a and b) depict the relationships between the normalized Young’s modulus and the Poisson’s ratio versus the strain for the DVM for different $\theta_2$. The conclusions drawn from Fig. 9 are generally observed in Fig. 10. It is noteworthy that DVM with smaller $\theta_2$ tends to buckle earlier as the load is increased because the L-beam is closer to the vertical beam. In those cases, the axial force in the L-beam becomes larger and cause buckling to occur sooner. Even at small values of strain, i.e., closer to the linear regime, $E_z$ for DVM with smaller $\theta_2$ can be large.

Fig. 11. Comparison between the theoretical and numerical prediction of mechanical properties for 2D DVM with different $l$ in $z$ direction (a) Normalized Young’s modulus (b) Poisson’s ratio.
and then decreases rapidly under large deflections. This result demonstrates the necessity to use nonlinear geometry modeling for the mechanical properties of DVM at large strain values (nonlinear regime).

The effects of the parameter $l$ on the mechanical properties of DVM under large deflections are shown in Fig. 11. Imposing the assumptions stated in Section 4, it has been proved that the Poisson’s ratio in the elastic regime is determined only by the angles of $\theta_1$ and $\theta_2$ of the S- and L-beams (Gao et al., 2018). The conclusion is still correct for the DVM with large $l$ under large deflections. However, there is a noticeable difference when $l$ is small, axial effects in both the S- and L-beams cannot be ignored, especially the S-beam. In other words, the assumption that the beams are sufficiently slender is incorrect. It is also seen that $E_z$ increases with increasing values of $l$. This observation can serve as an efficient method to enhance $E_z$ of DVM in designs.

5.2.2. Solutions in the x-direction

Because of the anisotropy microstructure of the DVM, it is necessary to analyze the mechanical properties in the other principal direction, the x-direction. Comparisons between the analytical and numerical predictions are shown in Figs. 12–14, where excellent agreement is observed at the selected values of the strain. Contrary to the results for $E_z$, the normalized Young’s modulus in x-direction $E_x$ decreases with increasing values of the strain. However, the Poisson’s ratio increases as the strain increases, while remaining negative. On the effects of the parameters $\theta_1$, $\theta_2$ and $l$ on the mechanical properties, $E_x$ increases while $\nu_{zx}$ decreases with increasing values of $\theta_1$, $\theta_2$. The dependence on $l$ is similar as in the z-direction. Moreover, no extremum points on the curves are observed, meaning that the beams of the DVM do not buckle.

Comparisons between the linear theoretical, nonlinear theoretical and numerical predictions of several DVM microstructures are summarized in Table 2. The summary shows that the nonlinear model agrees well with the numerical solutions, while the linear model is inaccurate.

5.3. Experimental verification

To experimentally validate the accuracy of the analytical model, a 2D DVM was manufactured using the additive printing technique. The
A prototype has five cells in the $x$-direction and four cells in the $z$-direction. The material parameters of each unit cell are: $\theta_1 = 60^\circ$, $\theta_2 = 30^\circ$, $l = 10$ mm and out-of-plane width $b = 18$ mm. The thickness of each of the L- and S-beams is 1 mm. The material for the prototype is ABS plastic with Young’s modulus $E_s = 2.2$ GPa, Poisson’s ratio $\nu_s = 0.39$ and mass density $\rho_s = 1.08$ g/cm$^3$. Quasi-static compression tests were conducted on a micro-controlled electronic universal testing machine (WDW3030) composed of a force transducer, an electronic dial gauge, a load head and base, as shown in Fig. 15. The 2D DVM prototype was placed on the platform and the load head compressed the top of the DVM microstructure at a speed of 50 mm/min. The values of the force load and the displacement along the compression direction were recorded. Fig. 16 compares the stress-strain curves of the DVM microstructure predicted by the theoretical, numerical and experimental methods. There is clearly a good agreement between these results. Thus, it is shown again that the analytical model proposed in this paper can predict the mechanical properties of the DVM microstructure under large deflections. The linear model, however, always underestimates the normalized Young’s modulus. When the strain is 0.15, the linear model predicts the stress as 24.1 MPa, which is 24.8% lower than the actual stress of 32.2 MPa. Therefore, to predict the mechanical properties of DVM microstructures accurately, it is necessary to adopt the nonlinear model in analyses and designs.

![Fig. 15. 2D DVM prototype under compression test.](image)

![Fig. 16. Comparison of the experimental, numerical and theoretical solutions.](image)

**Table 2**

Comparison of the normalized Young’s modulus of DVM microstructure in $x$ direction ($\epsilon = 0.1$).

<table>
<thead>
<tr>
<th>$\theta_1$ ($^\circ$)</th>
<th>$\theta_2$ ($^\circ$)</th>
<th>$l$ (mm)</th>
<th>Linear model (1e-3)</th>
<th>Nonlinear model</th>
<th>Numerical solution (1e-3)</th>
<th>Theoretical solution (1e-3)</th>
<th>Error (%)</th>
<th>Improvement $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>45</td>
<td>30</td>
<td>0.716</td>
<td>0.253</td>
<td>0.256</td>
<td>1.19</td>
<td>-64.66</td>
<td>-</td>
</tr>
<tr>
<td>60</td>
<td>15</td>
<td>20</td>
<td>0.102</td>
<td>0.074</td>
<td>0.073</td>
<td>1.35</td>
<td>-27.45</td>
<td>-</td>
</tr>
<tr>
<td>45</td>
<td>30</td>
<td>40</td>
<td>0.019</td>
<td>0.016</td>
<td>0.015</td>
<td>3.25</td>
<td>-15.79</td>
<td>-</td>
</tr>
</tbody>
</table>
6. Conclusion

In this work, nonlinear theoretical models of the 2D and 3D DVMs based on large beam deflection model are developed to predict the normalized Young’s modulus and Poisson’s ratio as functions of the strain in the principal directions. For 2D DVMs, finite element models are established to give the numerical solutions. It is found that there is a good agreement between the theoretical and numerical solutions. The parameters $\theta_1$, $\theta_2$ and $l$ that characterize the geometry of auxetic microstructure are shown to have significant effects on the mechanical properties of the DVMs under large deflections. For the mechanical properties of 2D DVM in the vertical direction, there is a critical strain which causes buckling of the L-beams. Beyond this critical strain, the normalized Young’s modulus $E_z$ decreases significantly while the Poisson’s ratio $\nu_{xy}$ increases. However, when the DVM with the same values of the parameters is compressed in the lateral direction, no buckling is observed. Also, the effects of $l$ on the mechanical properties of 2D DVM in the two directions are similar. To validate the accuracy of the nonlinear theoretical models for 2D DVMs under large deflections, prototypes are manufactured and quasi-static compression tests are conducted. There is excellent agreement between the nonlinear model theoretical and experimental results. It is also shown that the linear theoretical model always underestimates the stress levels at large strains. In summary, it is shown that the nonlinear theoretical model based on the large deflection beam model can accurately predict the mechanical properties of the 2D and 3D DVM in the principal directions. Our proposed model is thus useful for adoption in analyses and engineering designs for accurate predictions of the mechanical behavior at large strains.

Acknowledgments

This work was supported by China Scholarship Council (201606684004), which sponsored the first author as a visiting scholar at the University of Michigan, Ann Arbor, Michigan, USA for 2 years. This work was also supported by the National Natural Science Foundation of China (grant no 51675281) and the National Key Research and Development Program of China (2017YFC0803904).

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