Co-existing complexity-induced traveling wave transmission and vibration localization in Euler-Bernoulli beams

Xiangle Cheng, Lawrence A. Bergman, D. Michael McFarland, Chin An Tan, Alexander F. Vakakis, Huancai Lu

A S T R A C T

We analytically and numerically examine the dynamic behavior of an undamped, linear, uniform and homogeneous Euler-Bernoulli beam of finite length, partially supported in its interior by local, grounded, linear spring-dashpot pairs and subjected to a harmonic displacement at its pinned left boundary or at both its ends. The Euler-Bernoulli beam is known to be dispersive and, thus, to exhibit a non-constant relationship between frequency and wave number. Local dissipation due to the interior support results in a non-classically damped system and, consequently, mode complexity. An analytical framework is developed to examine the coexistence of propagating and standing harmonic waves in complementary regions of the beam for four distinct boundary conditions at the right end: pinned, fixed, free and linear elastic. We show that the system can be designed so that, for a particular input frequency and interior support location, nearly perfect spatial separation of traveling and standing waves can be achieved; the imperfection is shown to be caused by the non-oscillatory evanescent components in the solution. We further demonstrate that vibration localization is achieved by satisfying necessary and sufficient wave separation conditions, which correspond to frequency- and position-dependent support stiffness and damping values, and that linear viscous damping in the interior support, but not necessarily linear stiffness, is required to achieve the separation phenomenon and vibration localization.

1. Introduction

Understanding wave motion in structures is essential to the design of effective energy absorption strategies [1-5]. It is well known that complex modes of asynchronous motions between material points in a structure can arise due to the presence of non-classical damping in the distributed-parameter system [6,7]. Moreover, the non-classical damping enables the separation of standing waves and traveling waves that is fundamental for the realization of one-way energy propagation.

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Nomenclature

\begin{align*}
A & \quad \text{cross-sectional area of beam} \\
\hat{A}, \hat{A} & \quad \text{dimensional excitation amplitude, dimensionless excitation amplitude, } \hat{A} = \hat{A}/L \\
c & \quad \text{temporal scaling factor, } c = \sqrt{(EL)/(\rho A L^2)} \\
C_{11}, C_{12} & \quad \text{translational damping at } \hat{x} = \hat{x}_0 \text{ or } \hat{x} = \hat{x}_1, C_{11} = (C_{11} c_l^3)/(EL) \\
C_{21}, C_{22} & \quad \text{translational damping at } \hat{x} = \hat{x}_2, C_{22} = (C_{22} c_l^3)/(EL) \\
D_{in} & \quad (i = 1, 2, 3, n = 1, 2, 3, 4) \quad \text{complex coefficients of solutions in the subdomains} \\
E & \quad \text{Young’s modulus of beam} \\
E_{diss} & \quad \text{averaged energy dissipation by the local dampers} \\
E_{\text{in}} & \quad \text{averaged energy transfer into the system} \\
f_i, g_i, f_1, g_1, f_2, g_2 & \quad \text{constituent functions in the denominators of } D_{in} \\
f_{\text{in}}, S_{\text{in}}, f_{1, \text{in}}, g_{1, \text{in}}, f_{2, \text{in}}, g_{2, \text{in}} & \quad (i = 1, 2, 3, n = 1, 2, 3, 4) \quad \text{constituent functions in the numerator of } D_{in} \\
K_{11}, K_{12} & \quad \text{translational spring at } \hat{x} = \hat{x}_0 \text{ or } \hat{x} = \hat{x}_1, K_{11} = (K_{11} L^2)/(EL) \\
K_{21}, K_{22} & \quad \text{translational spring at } \hat{x} = \hat{x}_2 \text{ or } \hat{x} = L, K_{22} = (K_{22} L^2)/(EL) \\
K_{21}, K_{22} & \quad \text{rotational spring at } \hat{x} = L, K_{22} = (K_{22} L^2)/(EL) \\
\tilde{k}_{11}, \tilde{k}_{12} & \quad \text{complex stiffness, } \tilde{k}_{11} = k_{11} + j\omega c_{11}, \tilde{k}_{12} = k_{12} + j\omega c_{12} \\
L & \quad \text{length of beam} \\
l_2 & \quad \text{area moment of inertia} \\
t, \tau & \quad \text{dimensional time, dimensionless time, } \tau = ct \\
x & \quad \text{dimensionless axial coordinate along the centroidal axis of beam at equilibrium} \\
x_0, x_1, x_2 & \quad \text{dimensionless position of the translational viscoelastic support, } x_0 = \hat{x}_0/L, x_1 = \hat{x}_1/L, x_2 = \hat{x}_2/L \\
v(\cdot), V(\cdot) & \quad \text{deflection of beam, amplitude of the deflection} \\
\Omega, \omega & \quad \text{dimensional excitation frequency, dimensionless excitation frequency, } \omega = \Omega/c \\
\gamma & \quad \text{wavenumber} \\
\phi & \quad \text{spatial phase difference} \\
\kappa & \quad \text{function in the denominators of } k_{11}^* \text{ and } c_{11}^* \\
\phi, \psi & \quad \text{functions in the denominators of } D_{in}^* (i = 1, 2, n = 1, 2, 3, 4) \text{ for the pinned-clamped and pinned-free beam} \\
\rho_{E_1}, \rho_{E_2} & \quad \text{kinetic energy density, potential energy density} \\
\text{Re}(\cdot) & \quad \text{real part of a quantity} \\
(\cdot)^* & \quad \text{symbol denoting the necessary conditions for wave separation and vibration localization} \\
(\cdot) & \quad \text{symbol denoting certain dimensional functions and variables} \\
c.c & \quad \text{complex conjugate} \\
\| \cdot \|_2 & \quad \text{Euclidean norm} \\
\end{align*}
tuning. The beam is excited harmonically at one end and is attached to ground through a local linear spring-dashpot at an intermediate point. Four boundary conditions at the opposite end (pinned, clamped, free, and linear elastic) are considered in order to assess the effects of boundary conditions on spatial wave separation and vibration mode localization. An additional example of the pinned-pinned beam, which is subject to synchronous support motions at both ends and with two viscoelastic supports in its interior, is shown to verify that the vibration confinement strategy remains valid for the undamped Euler-Bernoulli beam in the presence of multiple spring-damper supports.

It is noted that complex modes in damped systems can also arise from gyroscopic, aerodynamic, and nonlinear effects [13,14]. Numerical discretization methods are generally applied to solve those problems. For example, Krenk [15,16] employed a state-space formulation, which required solving a complex, symmetric eigenvalue problem, to examine optimal damping and tuning of complex damped modes of taut cables and beams. For damped discrete structures often used for typical building models [17], an interpolation scheme was developed to find the free vibration solution of the damped system, approximated from the undamped and rigidly constrained eigensolutions. For the problem of a cantilever beam supported by local dampers [18], root locus and transfer function approaches were applied to predict the attainable damping ratio associated with the relative frequency shift of the structure with and without local dampers.

Damping-induced complex modes in structures have many important applications, especially in the design of passive supplemental energy dissipation devices [19,20]. These are well documented in the literature, with various analytical and numerical methods applied to solve the basic problem of beams with locally attached dampers [21–26]. Recently, non-reciprocal wave scattering in string-coupled oscillator systems and acoustic wave guides was investigated [27,28], which may have some application to studies of the human auditory system. It appears that the human basilar membrane of the cochlea works as a viscous damping element to passively receive the incident sound [29,30]. Wave absorption via the basilar membrane in the ear canal [31,32] is thus conceptually similar to generation of traveling waves in structures.

The remainder of this paper is organized as follows: Section 2 formulates the general boundary value problem, focusing on the pinned-pinned beam with harmonic base motion applied at the left support, and connected to ground at an interior point through a linear spring-dashpot. The other cases studied merely require a change in the boundary conditions at the right support or a modification of the equilibrium conditions at the attachment location, since the solution procedure remains the same. In Section 3, necessary and sufficient conditions for vibration localization through separation of the traveling waves from the standing waves are established for the pinned-pinned case. A discussion of complexity-induced vibration localization is also presented. Section 4 illustrates the major simulation results for the remaining three boundary cases—clamped, free, and linear elastic—with pertinent mathematical results summarized in the Appendix. The necessary formulations and simulation results for the beam with two viscoelastic supports are presented as well. Conclusions from this study are given in Section 5.

2. Problem formulation

Consider an undamped Euler-Bernoulli beam with \( \bar{x} \) denoting axial position along the centroidal axis at equilibrium. The beam is of length \( L \), Young’s modulus \( E \), cross-sectional area \( A \), area moment of inertia \( I_z \), and mass density \( \rho \). Four cases, representing the four different boundary conditions at \( \bar{x} = L \), are studied in this paper. In each case, the beam is pinned and given a harmonic displacement at its left end, \( \bar{x} = 0 \). A Voigt element, consisting of a translational linear spring \( K_1 \) and a viscous damper \( C_1 \), in parallel, connects the beam to ground at an interior point \( \bar{x}_a \). At its right end, \( \bar{x} = L \), the beam can be pinned, fixed, free, or connected to ground through linear translational and rotational springs, \( K_{12} \) and \( K_{22} \), respectively.

The transverse response of the beam is represented by the displacement, \( \bar{v}(\bar{x}, t) \), and the harmonic excitation applied to the boundary \( \bar{x} = 0 \) is \( \bar{v}(0, t) = \bar{A} \bar{e}^{i\Omega t} \), where \( \bar{A} \) is the amplitude and \( \Omega \) is the forcing frequency.

The non-dimensional parameters are defined as

\[
\bar{x} = \frac{\bar{x}}{L}, \quad \bar{v} = \frac{\bar{v}}{\bar{A}}, \quad \bar{A} = \frac{A}{L}, \quad \bar{c} = \sqrt{\frac{E L z}{\rho A L^2}}, \quad \bar{\tau} = ct, \quad \bar{\omega} = \frac{\Omega}{c},
\]

\[
k_{\bar{t}1} = K_1 L^3 \frac{E I_z}{E L_z}, \quad c_{\bar{t}1} = C_1 L^3 \frac{E I_z}{E L_z}, \quad k_{\bar{t}2} = K_{12} L^3 \frac{E I_z}{E L_z}, \quad k_{\bar{r}2} = K_{22} L^2 \frac{E I_z}{E L_z},
\]

where \( c \) is a temporal scaling factor with units sec\(^{-1}\). With this scaling, the coefficients of the governing differential equations are unity.

Employing the comma convention to represent partial differentiation, the normalized equations of motion of the system are

\[
\begin{align*}
v_{1,xxx}(x, \tau) + v_{1,tt}(x, \tau) = 0, & \quad 0 \leq x \leq x_a^- \\
v_{2,xxx}(x, \tau) + v_{2,tt}(x, \tau) = 0, & \quad x_a^- \leq x \leq 1, \quad \tau \geq 0,
\end{align*}
\]

and the initial conditions are assumed to be homogeneous such that \( v(x, 0) = v_{x}(x, 0) = 0 \). In what follows, the steady-state solutions corresponding to each of the three homogeneous boundary conditions (pinned, clamped, free), as well as the linear
elastic boundary condition at \( x = 1 \), are determined. The results for the pinned problem will first be derived with exposition of both the boundary value problem and steady-state solution. The formulations for each of the remaining three cases (fixed, free and linear elastic), which differ from the pinned example only by the boundary conditions at \( x = 1 \), are summarized in the Appendix.

The first case of interest, the pinned-pinned beam in normalized coordinates, is shown in Fig. 1. The boundary conditions are given by

\[
\nu_1(0, \tau) = \overline{A} e^{i \omega \tau}, \quad \nu_1(x, 0) = 0, \quad \nu_2(1, \tau) = 0, \quad \nu_2(x, 1) = 0, \quad (3)
\]

and the continuity and equilibrium conditions are

\[
\begin{align*}
\nu_1(x_0^-, \tau) &= \nu_2(x_0^-, \tau), \quad \nu_1(x_0^+, \tau) = \nu_2(x_0^+, \tau), \\
\nu_1(x_0^-, \tau) &= \nu_2(x_0^-, \tau), \quad \nu_2(x_0^+, \tau) - \nu_1(x_0^+, \tau) = -[k_{11} \nu_1(x_0^+, \tau) + c_{11} \nu_1(x_0^+, \tau)].
\end{align*}
\]

\[
(4)
\]

Assuming a steady-state solution of the form

\[

\nu_i(x, \tau) = V_i(x) e^{i \omega \tau}, \quad i = 1, 2,
\]

and substituting Eq. (5) into Eqs. (2)–(4) results in the boundary value problem

\[
\begin{align*}
V_i'(x) - \gamma^4 V_i(x) &= 0, \quad i = 1, 2, \\
V_1(0) &= \overline{A}, \quad V_1'(0) = 0, \quad V_2(1) = 0, \quad V_2'(1) = 0, \\
V_1(x_0^-) &= V_2(x_0^-), \quad V_1(x_0^+) = V_2(x_0^+), \quad V_1'(x_0^-) = V_2'(x_0^+), \quad V_2'(x_0^-) - V_1'(x_0^-) = -(k_{11} + j \omega c_{11}) V_1(x_0^-).
\end{align*}
\]

\[
(6)
\]

where \( \gamma^4 = \omega^2 \), and prime denotes ordinary differentiation with respect to \( x \). The steady-state amplitude of the displacement in each subdomain of the beam can then be written as

\[
\begin{align*}
V_1(x) &= D_{11} e^{\gamma x} + D_{12} e^{-\gamma x} + D_{13} e^{2\gamma x} + D_{14} e^{-2\gamma x}, \quad 0 \leq x \leq x_a^-, \\
V_2(x) &= D_{21} e^{\gamma(1-x)} + D_{22} e^{-\gamma(1-x)} + D_{23} e^{2\gamma(1-x)} + D_{24} e^{-2\gamma(1-x)}, \quad x_a^- \leq x \leq 1.
\end{align*}
\]

\[
(7)
\]

The leading two terms in each expression in Eq. (7) represent the harmonic or far-field components while the remaining two terms are the evanescent or near-field components in the two subdomain solutions.

Substitution of Eq. (7) into the boundary, continuity and equilibrium conditions of Eq. (6) results in the solutions for the eight complex coefficients, written in the compact form

\[
\begin{align*}
V_i(x) &= D_{11} e^{\gamma x} + D_{12} e^{-\gamma x} + D_{13} e^{2\gamma x} + D_{14} e^{-2\gamma x}, \quad 0 \leq x \leq x_a^- \\
V_2(x) &= D_{21} e^{\gamma(1-x)} + D_{22} e^{-\gamma(1-x)} + D_{23} e^{2\gamma(1-x)} + D_{24} e^{-2\gamma(1-x)}, \quad x_a^- \leq x \leq 1.
\end{align*}
\]
\[ D_{1n} = \frac{\tilde{k}_1 f_{1n}(\omega, \gamma, \chi_0) - g_{1n}(\omega, \gamma, \chi_0)}{\tilde{k}_1 f(\omega, \gamma, \chi_0) - g(\omega, \gamma, \chi_0)}, \]
\[ D_{2n} = \frac{\tilde{k}_1 f_{2n}(\omega, \gamma, \chi_0) - g_{2n}(\omega, \gamma, \chi_0)}{\tilde{k}_1 f(\omega, \gamma, \chi_0) - g(\omega, \gamma, \chi_0)}, \]  
\[ n = 1, \ldots, 4, \]  
\[ (8) \]
where \( \tilde{k}_1 = k_1 + j\omega \epsilon_1 \). The complex functions \( f_{1n}, f_{2n}, g_{1n}, g_{2n} \) are given below. Focusing on the left subdomain, \( 0 \leq x \leq \chi_0 \), the component functions in Eq. (8) are found to be
\[ f_{11} = A e^{-(1+j)\gamma} \left\{ (1-j) - (1+j) e^{2\gamma} + e^{(1+j)\gamma(2-\chi_0)} + e^{(1+j)\gamma \chi_0} + e^{2\gamma(1-j)\chi_0} - e^{2\gamma(1-\chi_0)} + je^{2\gamma(1-\chi_0)} - e^{2\gamma(1-\chi_0)} + je^{2\gamma(1-\chi_0)} \right\}, \]
\[ g_{11} = -8jA e^{-j\gamma^2 \sin(\gamma)}, \]
\[ f_{12} = A e^{(1+j)\gamma} \left\{ -(1-j) + (1+j) e^{-2\gamma} - e^{-(1+j)\gamma(2-\chi_0)} - e^{-(1+j)\gamma \chi_0} - e^{-2\gamma(1-j)\chi_0} + e^{-2\gamma(1-\chi_0)} - je^{-2\gamma(1-\chi_0)} + e^{-2\gamma(1-\chi_0)} - je^{-2\gamma(1-\chi_0)} \right\}, \]
\[ g_{12} = 8jA e^{-j\gamma^2 \sin(\gamma)}, \]
\[ f_{13} = A e^{(1+j)\gamma} \left\{ (1-j) + (1+j) e^{2\gamma} - je^{(1+j)\gamma(2-\chi_0)} - je^{(1+j)\gamma \chi_0} - je^{2\gamma(1-j)\chi_0} + je^{2\gamma(1-\chi_0)} - e^{2\gamma(1-\chi_0)} + je^{2\gamma(1-\chi_0)} - e^{2\gamma(1-\chi_0)} + je^{2\gamma(1-\chi_0)} \right\}, \]
\[ g_{13} = 8jA e^{-j\gamma^2 \sin(\gamma)}, \]
\[ f_{14} = A e^{(1+j)\gamma} \left\{ -(1-j) - (1+j) e^{-2\gamma} + je^{-(1+j)\gamma(2-\chi_0)} + je^{-(1+j)\gamma \chi_0} + je^{-2\gamma(1-j)\chi_0} - je^{-2\gamma(1-\chi_0)} + e^{-2\gamma(1-\chi_0)} - e^{-2\gamma(1-\chi_0)} + je^{-2\gamma(1-\chi_0)} \right\}, \]
\[ g_{14} = -8jA e^{-j\gamma^2 \sin(\gamma)}, \]
and the functions in the right subdomain, \( \chi_0^+ \leq x \leq 1 \),
\[ f_{21} = 4jA [\sin(\gamma \chi_0) + \sinh(\gamma \chi_0)] [\sinh(\gamma(1-\chi_0))], \]
\[ g_{21} = 8jA \gamma^2 \sin(\gamma), \]
\[ f_{22} = 4jA [\sin(\gamma \chi_0) + \sinh(\gamma \chi_0)] [\sin(\gamma(1-\chi_0))], \]
\[ g_{22} = -8jA \gamma^2 \sin(\gamma), \]
\[ f_{23} = 4A [\sin(\gamma \chi_0) + \sinh(\gamma \chi_0)] [\sin(\gamma(1-\chi_0))], \]
\[ g_{23} = -8jA \gamma^2 \sin(\gamma), \]
\[ f_{24} = 4jA [\sin(\gamma \chi_0) + \sinh(\gamma \chi_0)] [\sin(\gamma(1-\chi_0))], \]
\[ g_{24} = 8jA \gamma^2 \sin(\gamma). \]
\[ (9) \]
The functions \( f \) and \( g \) in the denominator are
\[ f = 16 \{ \sin(\gamma) \cosh(\gamma) \sinh^2(\gamma \chi_0) + \sin(\gamma) \{ \sin(\gamma \chi_0) \sin(\gamma(1-\chi_0)) \} \} - \sin(\gamma) \sinh(\gamma \chi_0) \cosh(\gamma \chi_0) \}, \]
\[ g = -32 \gamma^2 \sin(\gamma) \sinh(\gamma). \]
\[ (10) \]
\[ (11) \]
3. Vibration localization in the pinned-pinned beam resulting from complexity induced by local damping at the interior support

A solution to the boundary value problem defined previously is sought such that the vibration is localized to the segment of the beam to the right of the viscoelastic interior support; i.e., \( \chi_0^+ \leq x \leq 1 \). This ideally requires that energy input via a harmonic displacement at \( x = 0 \) be transmitted from left to right by a traveling wave, pass through the interior support at \( x = \chi_0 \) without reflection, and continue on to the pinned support at \( x = 1 \), where it is perfectly reflected back toward the interior support with no energy passing through to the source. To achieve ideal vibration localization at steady state, the right-moving and left-moving waves at the right of the interior support must be at the same wavenumber \( \gamma \), such that interference between them creates a standing wave, with excess energy dissipated at the interior support. This ideal behavior can be observed in, for example, non-dispersive systems such as strings and rigid circular acoustic ducts \([7,9,11,12]\). However, the Euler-Bernoulli beam is a dispersive system, exhibiting non-oscillatory, rapidly decaying evanescent waves confined to regions near the supports. Thus, perfect localization is generally not possible in such systems, and the regions of the beam where the evanescent modes contribute exhibit boundary layers of complexity which locally distort the trivial phase relations in the standing wave.
Imposing the boundary conditions of the prescribed displacement and zero bending moment at \( x = 0 \) leads to the harmonic and evanescent conditions, respectively

\[
D_{11} + D_{12} = \frac{\bar{A}}{2}, \quad D_{13} + D_{14} = \frac{\bar{A}}{2}
\] (12)

Then, a purely right-moving traveling wave in the subdomain \( 0 \leq x \leq x_a \) can be achieved by setting the complex coefficient corresponding to the left-moving harmonic wave in that subdomain to zero; i.e., \( D_{11} = 0 \). Combining this requirement with the two conditions of Eq. (12) gives

\[
D_{12} + D_{13} + D_{14} = \bar{A},
\]

which is a necessary condition for localization for all the boundary cases to be examined.

Employing the formulae in Eq. (8) and solving the resulting equation yields a set of specific values of \( \bar{k}_{11} \), denoted by \( \bar{k}_{11}^* \), that represent the solutions to the complex equation

\[
\bar{k}_{11}^* = k_{11}^* + j \omega c_{11}^* = \frac{g_{11}(\omega, \gamma, x_a)}{f_{11}(\omega, \gamma, x_a)}. \tag{14}
\]

This can be reduced to two real equations for the interior linear support stiffness and damping values that localize vibration at specific values of frequency \( \omega \) and interior support location \( x_a \). The explicit expressions for \( k_{11}^* \) and \( c_{11}^* \) are given by

\[
k_{11}^* = \frac{2\gamma^3}{\kappa} \left( \sinh(2\gamma x_a) - \sin[2\gamma(1 - x_a)] - \sin[\gamma(2 - x_a)] \sinh(2\gamma) \sinh(\gamma x_a) \right.
\]
\[
+ \sin(\gamma x_a) \sinh(2\gamma) \sin(\gamma x_a) + 2 \sin(\gamma x_a) \sinh^2(\gamma x_a)
\]
\[
+ \cosh(2\gamma) \left( \sin[2\gamma(1 - x_a)] - \sin(2\gamma x_a) \right)
\]
\[
+ 2 \cosh(\gamma x_a) \sinh^2(\gamma) (\sin[\gamma(2 - x_a)] - \sin(\gamma x_a)) \right),
\]

\[
c_{11}^* = \frac{8\gamma \sinh(\gamma) \sin[\gamma(1 - x_a)]}{\kappa} \left( \sinh(\gamma) \sin[\gamma(1 - x_a)] + \sin[\gamma(1 - x_a)] \sin(\gamma) \right), \tag{16}
\]

where

\[
\kappa = \delta_1^2 + \delta_2^2 = \left( \sinh(\gamma) \sin[2\gamma(1 - x_a)] - \{ \sin(\gamma x_a) + 2 \sin(\gamma x_a) - \sin[\gamma(2 - x_a)] \} \sin(\gamma(1 - x_a)) \right)^2
\]
\[
+ \left( 2 \sinh(\gamma) \sin^2[\gamma(1 - x_a)] + \{ \cos(\gamma x_a) - \cos[\gamma(2 - x_a)] \} \sin(\gamma(1 - x_a)) \right)^2. \tag{17}
\]

Critical values of the stiffness and damping of the interior viscoelastic support leading to spatial separation of traveling and standing waves to achieve vibration localization to the right of the interior support are plotted in Fig. 2. In Fig. 2(a), the necessary values of support stiffness and damping are plotted versus frequency with the location of the interior support fixed at \( x_a = 0.4 \), while in Fig. 2(b) the corresponding values are shown versus support location \( x_a \) for a fixed frequency \( \omega = 100\pi \). It is noted that, in both figures, interior support stiffness and damping values can go to zero at irregularly spaced points. However, this is not acceptable for damping, as complexity due to local damping in the interior support is required to achieve the spatial wave separation phenomenon leading to vibration localization. These points must, in fact, be excluded to guarantee that the given conditions are both necessary and sufficient. Since \( c_{11}^* \) must be strictly non-negative, and the function \( \kappa \) in Eq. (17) can be shown to be finite since it is continuous and defined in a compact metric space \([33]\), the zeros of \( c_{11}^* \) must result from one or both of the bracketed terms in the numerator of Eq. (16). The first term gives \( \sin[\gamma(1 - x_a)] = 0 \), the nontrivial solutions of which are \( \gamma_r = r \frac{\pi}{1 - x_c} \), \( r = 1, 2, \ldots \), and the corresponding frequencies are \( \omega_r = r^2 \left( \frac{\pi}{1 - x_c} \right)^2 \), \( r = 1, 2, \ldots \). The second term is given by

\[
\sinh(\gamma) \sin[\gamma(1 - x_a)] + \sin(\gamma(1 - x_a)) \sin(\gamma) = 0,
\]

which must be solved numerically. For the parameters employed to obtain Fig. 2(a), the first equation yields the set of frequencies \( \omega_r = 27.4156, 109.6623, 246.7401, 438.6491, \ldots \), while the second gives \( \omega_c = 25.2319, 110.1228, 246.7400, 438.6352, \ldots \), verifying the values in the figure. Note that the trivial frequencies (\( \omega_r = \omega_c = 0 \)), excluded from the aforementioned solutions, can be used to interpret the static deflection of the beam under a constant point load. For the parameters used to obtain Fig. 2(b), the first equation yields the set of positions \( x_{ar} = 0.1138, 0.2910, 0.4683, 0.6455, 0.8228, \ldots \), while the second
gives \( x_{da} = 0.1199, 0.2907, 0.4683, 0.6455, 0.8228, \ldots \), verifying the values in the figure. The zeros (roots of \( c_{11}^* = 0 \)) indicate the boundaries of the regions in which spatial wave separation cannot be achieved. These regions are closely spaced but do not coincide even with very fine numerical resolution in the root computations. Note that the two sets of frequencies \( \omega_r \) and \( \omega_c \) increase alternately at a fixed attachment location \( x_a = 0.4 \), and the two sets of positions \( x_{da} \) and \( x_{ba} \) increase alternately at a fixed excitation frequency \( \omega = 100\pi \). When the viscoelastic support moves close to the pinned right end, satisfaction of Eqs. (15) – (17) requires the spring stiffness \( k_{11}^* \) to be negative; thus, spatial wave separation in that region is not physically realizable, as indicated in Fig. 2(b). Since strong constraint forces arising from the boundaries are imposed on the passive attachment, “dead zones” occur at the beam ends where a support cannot be placed and induce separation.

Imposing the pinned right boundary condition at \( x = 1 \) leads to the relations

\[
D_{21}^* = -D_{22}^*, \quad D_{23}^* = -D_{24}^*.
\]

(19)

The steady-state solution can then be written as

\[
v_1(x, \tau) = \frac{\bar{A} e^{-j\gamma x} + \bar{A} e^{-\gamma x} + 2D_{13}^* \sin(\gamma x)}{\sinh(\gamma x) e^{j\bar{A}r}}, \quad 0 \leq x \leq x_a^*,
\]

\[
v_2(x, \tau) = 2 \left\{ jD_{21}^* \sin[\gamma(1-x)] + D_{23}^* \sinh[\gamma(1-x)] \right\} e^{j\bar{A}r}, \quad x_a^* \leq x \leq 1,
\]

(20)

where

\[
D_{13}^* = \frac{\bar{A} \sinh[\gamma(1-x_a)] e^{-j\gamma x} + e^{-\gamma x} \sin[\gamma(1-x_a)]}{\sinh(\gamma x) \sin[\gamma(1-x_a)]},
\]

\[
D_{21}^* = \frac{\bar{A}}{4j} \frac{e^{-j\gamma x_a}}{\sin[\gamma(1-x_a)]}, \quad D_{23}^* = \frac{\bar{A}}{4} \frac{\sin[\gamma(1-x_a)] - \sinh(\gamma x_a) e^{-j\gamma x}}{\sinh(\gamma) \sin[\gamma(1-x_a)]}.
\]

(21)

The function \( \sin[\gamma(1-x_a)] \) appearing in the numerator of \( c_{11}^* \) also appears in the denominator of the coefficients \( D_{13}^* \) and \( D_{23}^* \). Thus, the magnitudes of \( D_{13}^* \) and \( D_{23}^* \) go to infinity at the aforementioned set of frequencies \( \omega_r \), as shown in Fig. 3. This explains the mechanism by which the beam becomes resonant when the interior support damping becomes zero. Since \( k_{11}^* \) and \( c_{11}^* \) cannot approach zero simultaneously, the frequencies \( \omega = \omega_r \) are associated with the natural frequencies of a pinned-pinned beam with an interior elastic support of stiffness \( k_{11}^* (\omega_r, x_a) \). Interestingly, the frequency equation \( \sin[\gamma(1-x_a)] = 0 \) corresponds to that of a pinned-pinned beam of length \( (1-x_a) \), which is conceptualized as a beam of unit length with a very stiff elastic support at \( x = x_a \), which is consistent with Fig. 2. Note that \( D_{13}^* \) and \( D_{23}^* \) tend to zero with increasing \( \gamma \), ensuring that \( v_1(x, \tau) \) is dominated by the outgoing traveling wave and \( v_2(x, \tau) \) by the standing wave. This is demonstrated by the results.
of a simulated case, shown in Fig. 4, with parameters \( \omega = 100\pi, \ x_a = 0.4, \ \bar{A} = 1, \ k_{t1} = 2377.43 \) and \( c_{t1} = 39.00 \). To understand energy propagation associated with the wave separation dynamics, and adopting the previously defined normalizations, the kinetic and potential energy densities are defined, respectively, as
\[ \rho_{E_i} = \frac{1}{2} \text{Re}[\psi_t(x, \tau)]^2, \quad \rho_{E_e} = \frac{1}{2} \text{Re}[\psi_{xx}(x, \tau)]^2, \]

where \( \text{Re}[\cdot] \) represents the real part of the quantity in the square brackets.

Examination of the displacement, phase and energy density evolution plots, in Fig. 4(a) – 4(d), reveals spatial separation, with the traveling wave to the left and standing wave to the right of the interior support. The traveling wave is characterized by nearly linear variation of phase, with mild distortions near the source and interior support, while the standing wave is characterized by trivial phase differences everywhere except the neighborhood of the interior support. These variations are due to boundary layers of complexity induced by the evanescent components of the solution arising from dispersion. This can be further demonstrated by decomposing the harmonic and evanescent components of the response, as shown in Fig. 5.

Figures 5(a), (b) and (e) do, in fact, confirm that perfect separation of traveling and standing waves and complete localization of vibration can be achieved when considering only the harmonic component, while Figs. 5(c), (d) and (f) define

---

**Fig. 5.** Decomposition of the harmonic and evanescent parts of the responses in Fig. 4(a) – 4(c): (a) harmonic component of displacements; (b) spatial phase of the harmonic component normalized by \( \pi \); (c) evanescent component of displacements; (d) spatial phase of the evanescent component normalized by \( \pi \); (e) harmonic component of kinetic energy density; (f) evanescent component of the kinetic energy density. The vertical dashed lines identify the location of the interior linear viscoelastic support.
precisely the deviations from the ideal that appear as boundary layers of complexity at the source and interior support in the total solution of Fig. 4.

From Fig. 2, it is apparent that either the interior stiffness or damping can be zero with the variation of external excitation frequency $\omega$ and attachment location $x_a$. The specific regions, corresponding to zero damping, should be interpreted as insufficient conditions for realizing wave separation. However, for regions of zero stiffness, the dashpot alone is sufficient to generate the separation phenomenon, which is well known since pure damping-induced traveling waves have been observed in the nondispersive string and bar [7,8] wherein the local attachment is placed at an otherwise free boundary. At $\omega = 100\pi$ and $x_a = 0.60$, the simulation results with pure damping $c_{11}' = 70.51$ are given in Fig. 6(a) and Fig. 6(b), from which we find that wave separation is still attainable. When the support is moved to $x_a = 0.65$, Eq. (14) requires the system to be undamped with $k_{11}' = 22273.40$, which results in a global resonant mode with trivial spatial phase distribution, as seen from Fig. 6(c) and (d).

4. Simulation results for additional examples

The necessary and sufficient conditions for spatial separation of traveling waves and standing waves, and the specific vibration localization dynamics, in a pinned-pinned Euler-Bernoulli beam connected to ground through a local linear spring-damper have been studied in detail in Sections 2 and 3. In this section, we extend the discussion to Euler-Bernoulli beams for three additional boundary conditions at the right end: clamped, free, and linear elastic, to understand the effects of non-dissipative boundary conditions on the vibration localization. However, with no need to duplicate the mathematical derivations step by step, the main formulations and equations are summarized in the Appendix, illustrating the basic differences from the pinned-pinned problem. In fact, the dynamics in terms of wave separation as well as vibration localization are independent of the boundary conditions to be considered. The key simulation results are shown and discussed in the following subsections.

---

Fig. 6. Steady-state response of the pinned-pinned beam with an interior viscoelastic support at $\omega = 100\pi$ and $\Lambda = 1$: (a) displacements and (b) spatial phase of the displacements normalized by $\pi$ with $x_a = 0.60$. $K_{11}' = 0.00$ and $c_{11}' = 70.51$; (c) displacements and (d) spatial phase of the displacements normalized by $\pi$ with $x_a = 0.65$. $K_{11}' = 22273.39$ and $c_{11}' = 0.00$. The vertical dashed lines identify the location of the interior support.
4.1. Clamped boundary at \( x = 1 \)

The formulation in normalized coordinates with a clamped boundary condition at \( x = 1 \) differs from the previous pinned-pinned example only in the latter of the two boundary conditions,

\[
v_2(1, \tau) = 0, \quad v_{2,x}(1, \tau) = 0.
\]  

(23)

Following the same solution procedure as for the previous example, the boundary value problem, similar to Eq. (6), can be obtained and solved. For this case, the complex coefficients \( D_{i,n} \) \((i = 1, 2, \quad n = 1, \ldots, 4)\) from the solution of Eq. (7) can be employed to find the necessary and sufficient conditions for vibration localization, as given by the specific values of \( k_{t1}^* \) and \( c_{t1}^* \) in Eqs. (A.1) – (A.3). Provided that the conditions for vibration localization are established, the system responses can be simplified to Eqs. (A.4) – (A.6). Figure 7 displays the variation of the critical values of stiffness and damping with excitation frequency \( \omega \) and support location \( x_a \) necessary for wave separation and vibration localization, based on Eqs. (A.1) – (A.3). The dips in the plots of \( c_{t1}^* \) versus \( \omega \) and \( x_a \) in Figs. 7(a) and (b) can be found by numerically solving the expression \( c_{t1}^* = 0 \), as was done in Section 3.

Note that, when the support location \( x_a \) is fixed and \( c_{t1}^* \) is equal to zero, the roots \((\omega = \omega_r)\) denote the natural frequencies of a pinned-clamped beam on an interior elastic support with stiffness \( k_{t1}^*(\omega_r, x_a) \). Comparing the dips in the curves of \( c_{t1}^* \) versus \( \omega \) in Fig. 2(a) and Fig. 7(a), the natural frequencies are higher in the clamped case as it is stiffer than the pinned case. In terms of wave separation, given a fixed excitation frequency \( \omega \), Fig. 7(b) shows that the effective region of support location shrinks compared to that of the previous case given in Fig. 2(b). The clamped boundary imposes a stronger constraint on the system so that energies carried by reflected waves from the boundaries leak through the interior support as the support location moves toward the ends of the beam.

According to Eqs. (A.4) – (A.6), with the excitation frequency at \( \omega = 100\pi \) and the excitation amplitude at \( A = 1 \), proper selection of the spring constant and damping coefficient in the interior support results in separation of traveling and standing waves for \( x_a = 0.40 \), as shown in Figs. 8(a) and (b), and globally resonant vibration for \( x_a = 0.60 \), as shown in Figs. 8(c) and (d). Examination of Figs. 8(a) and (b) reveals that, due to evanescent effects, imperfect traveling waves are generated in the vicinity of the source and the interior support. Parameter values in the caption of Figs. 8(a) and (b) differ from those in Figs. 4(a) and (b) only because the fixed right boundary makes the beam stiffer. As a result, Eqs. (A.4) – (A.6) require the parameters of the interior support to be larger in order to realize wave separation.

![Fig. 7. Depiction of parameter values \( k_{t1}^* \) and \( c_{t1}^* \) for wave separation in the pinned-clamped beam: (a) separation conditions versus excitation frequency \( \omega \) for attachment location \( x_a = 0.4 \); (b) separation conditions versus attachment location \( x_a \) for excitation frequency \( \omega = 100\pi \).](image-url)
4.2. Free boundary at \( x = 1 \)

We repeat the problem formulation and solution scheme outlined in Section 4.1 to consider a free boundary at the right end. The boundary conditions are

\[
v_{2,xx}(1, \tau) = 0, \quad v_{2,xxx}(1, \tau) = 0.
\]  

(24)

The major formulations that consist of the necessary conditions and the resulting localized responses are highlighted in the Appendix, as given by Eqs. (A.7) – (A.9) and Eqs. (A.10) – (A.12), respectively. In particular, Fig. 9 shows the plots of the separation conditions.

Comparing Eqs. (A.7) – (A.9) and Eqs. (A.1) – (A.3), the separation conditions for the free-boundary case differ only slightly from those of the clamped case. Therefore, Fig. 9(a) nearly coincides with Fig. 7(a) when the right boundary is altered from clamped to free. As mentioned in the pinned-pinned example, the dips in the plots of \( c_{t1}^* \) versus \( \omega \) indicate where global resonances occur. It is known that the natural frequencies of a pinned-clamped beam are the same as those of the pinned-free beam with the exception of the rigid-body rotation (trivial natural frequency) in the pinned-free case \([3,5]\). Therefore, examination of prefect coincidence of natural frequencies in the two standard cases can be used to interpret the complexity-induced dynamics, in particular for the similarity of the conditions for wave separation and vibration localization.

Note that Fig. 9(b) indicates that a physical combination of \( k_{t1}^* \) and \( c_{t1}^* \) is realizable when the viscoelastic interior support moves to the free right boundary; i.e., \( x_0 = 1 \). In this situation, the necessary conditions for eliminating the left-moving harmonic wave \((D_{11} = 0)\) are found to be
When the excitation frequency is chosen as \( \omega = 100\pi \), Eq. (25) gives \( k_{11}^* = 2784.16 \) and \( c_{11}^* = 8.86 \), resulting in nearly pure traveling waves distributed along the entire beam. This special case demonstrates that boundary impedance matching nearly enables the input energy to be transmitted in one direction [7,8], which allows the Euler-Bernoulli beam to act as a one-dimensional waveguide. If the beam is sufficiently long to enable the evanescent waves to decay rapidly except at the boundaries, Fig. 10(a) indicates that nearly pure traveling waves can be observed along the beam.

Based on Eqs. (A.10) – (A.12), Fig. 11 depicts simulation results for two cases. In Fig. 11(a) the wave separation that takes place for strictly positive damping is demonstrated, and in Fig. 11(b), the global resonant mode that results in the case of zero damping is found. The parameter settings used in these simulations are shown in the caption of Fig. 11.
The beam vibration problem with a linear elastic boundary is important for developing useful guidelines for passive control strategies. The compliant boundary condition can be employed to simulate any of the three cases already discussed by setting the translational spring stiffness $k_t^2$ and rotational spring stiffness $k_r^2$ to their appropriate limiting values. Employing the normalized parameters in Eq. (1) for $k_t^2$ and $k_r^2$, the compliant boundary conditions are shown to be

$$ v_{2,x}(1, \tau) = -k_{t2} v_{2,xx}(1, \tau), \quad v_{2,xxx}(1, \tau) = k_{r2} v_2(1, \tau). \quad (26) $$

However, the wave separation conditions with respect to $k_{t1}^*$ and $c_{r1}^*$, along with the steady-state responses $v_1(x, \tau)$ and $v_2(x, \tau)$, are quite lengthy; therefore, we present only the key results. By means of numerical extraction of the real and imaginary parts of $k_{t1}^*$, the curves describing the necessary and sufficient conditions for the wave separation and vibration localization are shown in Fig. 12.

Continuing, the displacement and phase plots are given in Figs. 13(a) – (d). The linear elastic support at the right end is non-dissipative and, thus, provides total energy reflection. Exclusive of the neighborhood of the interior support, the right portion of the beam exhibits a purely standing-wave motion, which is characterized by trivial phase difference, as can be seen from Figs. 13(a) and (b).

However, when separating the harmonic and evanescent components from the total displacements and spatial phase in Figs. 13(a) and (b), the individual displacement components fail to satisfy the boundary condition at the right end. Focusing on the spatial phase for the far-field component and near-field component at $x = 1$, and noting that $\phi_{\text{far}} = \phi_{\text{near}} = 0.7432 \pi$ while $\phi = -1.2568 \pi$, we observe that the in-phase motion at the right end indicates that the displacement components, instead of being canceled, are superposed to yield the total displacement, as can be seen from Fig. 13(a), Fig. 14(a) and Fig. 14(c).
Fig. 12. Plots of the parameter values $k_1^*$ and $c_1^*$ for wave separation in the compliant case with $k_2 = k_2' = 2000$: (a) separation conditions versus excitation frequency $\omega$ for attachment location $x_a = 0.4$; (b) separation conditions versus attachment location $x_a$ for excitation frequency $\omega = 100\pi$.

Fig. 13. Steady-state response of the beam with an interior viscoelastic support and a compliant support at the right end, with $\omega = 100\pi$ and $\bar{\lambda} = 1$: (a) displacements and (b) spatial phase of the displacements normalized by $\pi$ with $x_a = 0.4$, $k_1^* = 10636.44$ and $c_1^* = 10.43$; (c) displacements and (d) spatial phase of the displacements normalized by $\pi$ with $x_a = 0.72$, $k_1^* = 22342.51$ and $c_1^* = 0.00$. The vertical dashed lines identify the location of the interior support.
While the previous examples were composed of a beam with a single viscoelastic support, it is worthwhile to further consider the use of two viscoelastic supports to limit the flexural waves to a region of the beam. We examine a pinned-pinned Euler-Bernoulli beam attached to two translational spring-dashpot pairs to verify that the flexural waves become standing.
waves locally confined between the dampers and traveling waves are directed from the energy sources towards the dampers. This is similar to the findings for the non-dispersive string system in Ref. [9]. In non-dimensional coordinates, the attachments are placed asymmetrically at the locations \( x = x_1 \) and \( x = x_2 \). Both ends of the beam are subject to identical harmonic motions \( \bar{A}\bar{w}\Delta \), as depicted in Fig. 15.

The system is partitioned into three segments by the local springs and dampers. The normalized equations of motion in each subdomain are formulated similarly as in the pinned-pinned case with a single spring-dashpot, resulting in

\[
\begin{align*}
v_{1,xxx}(x, \tau) + v_{1,\tau\tau}(x, \tau) &= 0, \quad 0 \leq x \leq x_1^1 \\
v_{2,xxx}(x, \tau) + v_{2,\tau\tau}(x, \tau) &= 0, \quad x_1^1 \leq x \leq x_2^2, \quad \tau \geq 0, \\
v_{3,xxx}(x, \tau) + v_{3,\tau\tau}(x, \tau) &= 0, \quad x_2^2 \leq x \leq 1
\end{align*}
\tag{27}
\]

where the resulting subsystems are indexed from left to right. The boundary conditions are given by

\[
v_1(0, \tau) = \bar{A}e^{i\omega t}, \quad v_{1,xx}(0, \tau) = 0, \quad v_3(1, \tau) = \bar{A}e^{i\omega t}, \quad v_{3,xx}(1, \tau) = 0,
\tag{28}
\]

and the continuity and equilibrium conditions at \( x = x_1 \) and \( x = x_2 \) are

\[
\begin{align*}
v_1(x_1^1, \tau) &= v_2(x_1^1, \tau), \quad v_{1,xx}(x_1^1, \tau) = v_{2,xx}(x_1^1, \tau), \\
v_3(x_2^2, \tau) &= v_2(x_2^2, \tau), \quad v_{3,xx}(x_2^2, \tau) = v_{2,xx}(x_2^2, \tau)
\end{align*}
\tag{29}
\]

Substituting the steady-state assumptions \( v_i(x, \tau) = V_i(x)e^{i\omega t} \) \((i = 1, 2, 3)\) into Eqs. \((27) - (29)\) gives the boundary value problems

\[
\begin{align*}
V_1(x) - \gamma^4 V_3(x) &= 0, \quad i = 1, 2, 3, \\
V_1(0) = \bar{A}, \quad V_1(0) = 0, \quad V_3(1) = \bar{A}, \quad V_3(1) = 0, \\
V_1(x_1^1) &= V_2(x_1^1), \quad V_{1,xx}(x_1^1) = V_{2,xx}(x_1^1), \\
V_2(x_2^2) &= V_3(x_2^2), \quad V_{2,xx}(x_2^2) = V_{3,xx}(x_2^2), \\
V_3(x_2^2) &= V_2(x_2^2)
\end{align*}
\tag{30}
\]

which lead to the steady-state amplitudes of the displacements in the form

\[
\begin{align*}
V_1(x) &= D_{11}e^{j\omega_1 x} + D_{12}e^{-j\omega_1 x} + D_{13}e^{j\omega_2 x} + D_{14}e^{-j\omega_2 x}, \quad 0 \leq x \leq x_1^1, \\
V_2(x) &= D_{21}e^{j\omega_1 x} + D_{22}e^{j\omega_2 x} + D_{23}e^{j\omega_3 x} + D_{24}e^{j\omega_3 x}, \quad x_1^1 \leq x \leq x_2^2, \\
V_3(x) &= D_{31}e^{j\omega_1 (1-x)} + D_{32}e^{j\omega_2 (1-x)} + D_{33}e^{j\omega_3 (1-x)} + D_{34}e^{j\omega_3 (1-x)}, \quad x_2^2 \leq x \leq 1
\end{align*}
\tag{31}
\]

The amplitude of the displacement in each subdomain of the beam can then be written in the compact form

\[
D_{in} = \frac{\hat{k}_{i1} D_{f1, in} + \hat{k}_{i2} D_{f2, in} + \hat{g}_{i1} g_{1, in} + \hat{g}_{i2} g_{2, in}}{\hat{k}_{i1} \hat{k}_{i2} f_1 + \hat{k}_{i1} g_{1} + \hat{k}_{i2} g_{2} + \hat{g}_{2}}, \quad i = 1, \ldots, 3, \quad n = 1, \ldots, 4,
\tag{32}
\]

where \( \hat{k}_{i1} = k_{i1} + j\omega C_{i1} \) and \( \hat{k}_{i2} = k_{i2} + j\omega C_{i2} \). Note that upon simplification \( f_1, f_{1, in}, f_2, f_{2, in}, g_1, g_{1, in}, g_2, g_{2, in} \) are defined as functions of \( \omega, \gamma, x_1 \) and \( x_2 \), similar to the linear elastic case in Section 4.3.

The conditions \( D_{11} = 0 \) and \( D_{33} = 0 \) are enforced to produce the steady-state solution that admits nearly pure traveling wave propagation from the energy sources to the dampers. These two conditions lead to expressions for the stiffness and damping of the individual attachments,

\[
\hat{k}_{i1} = -\frac{\hat{k}_{i1}^* f_{1,11} + g_{2,11}}{f_{1,11} + g_{1,11}} \quad \text{and} \quad \hat{k}_{i2} = -\frac{\hat{k}_{i2}^* g_{1,31} + g_{2,31}}{f_{1,31} + g_{2,31}}.
\tag{33}
\]

Tuning the parameters of the attachments for traveling wave realization corresponds to solving Eq. \((33)\) simultaneously, which reduces to solving a quadratic equation for each complex stiffness. Physically, the parameter values of each attachment should be positive so energy can be dissipated. We choose the solutions where both \( k_{i1}^*, c_{i1}^*, k_{i2}^* \) and \( c_{i2}^* \) are non-negative and plot the these parameters versus external frequency \( \omega \) for one asymmetric case, \( x_1 = 0.30 \) and \( x_2 = 0.80 \), as shown in Fig. 16. Different from the previous cases with only one interior support, given the fixed attachment locations, Fig. 16 shows that the effective frequency range for wave separation is narrower, which is understandable since the enforced conditions for wave separation are more restrictive. In addition, the two damping coefficients do not go to zero simultaneously over the
frequency range of interest, which indicates that the global vibration localization phenomenon cannot be realized during this parameter tuning process.

When the excitation frequency is specified as $\omega = 700$, the parameter values of the attachments are found to be

$$
\begin{align*}
&k^*_1 = 1.03 \times 10^5, \quad c^*_1 = 102.74, \quad k^*_2 = 2.66 \times 10^5, \quad c^*_2 = 84.41,
\end{align*}
$$

leading to the development of traveling waves in the beam sub-spans $0 \leq x \leq x_1$ and $x_2 \leq x \leq 1$. Note that the spatial phase of displacement $v_2$ is non-trivial in such a case, involving small complexity and giving modulated standing waves between the spring-damper supports; while $v_1$ and $v_3$ are dominated by nearly linear phases, with variations only at the boundaries, a result of the near-field evanescent waves. Figures 17(a) and

![Fig. 16. Depiction of the non-negative parameter values of $k^*_1$, $c^*_1$, $k^*_2$ and $c^*_2$ versus frequency $\omega$ for wave separation in the pinned-pinned beam for the support placement $x_1 = 0.30$ and $x_2 = 0.80$.]

![Fig. 17. Steady-state response of the pinned-pinned beam with two interior linear viscoelastic supports: (a) displacements; (b) spatial phase of the displacements normalized by $\pi$. Parameters are $\omega = 700$, $x_1 = 0.30$, $x_2 = 0.80$. $k^*_1 = 1.03 \times 10^5$, $c^*_1 = 102.74$, $k^*_2 = 2.66 \times 10^5$ and $c^*_2 = 84.41$. The vertical dashed lines identify the locations of the interior supports.](image)
The applied displacements $\psi(0, \tau) = A e^{b \tau t}$ and $\psi(1, \tau) = A e^{b \tau t}$ on the boundaries of the beam are the only energy inputs to the system. The energies are re-distributed to satisfy the condition for spatial separation of traveling and standing waves in the presence of properly tuned coefficients of the damping. In complex notation, we employ the averaged power flow per cycle \cite{[5]} to calculate the energy transfer from $x = 0$ and $x = 1$, giving

$$E_{\text{in}} = \frac{1}{2} Re\{c.c.[\psi_{xx}(x = 0, \tau)]v_T(x = 0, \tau) + \frac{1}{2} Re\{c.c.[\psi_{xx}(x = 1, \tau)]v_T(x = 1, \tau)\}, \quad (34)$$

while the averaged energy dissipation per cycle by the dampers at $x = x_1$ and $x = x_2$ is expressed by

$$E_{\text{diss}} = \frac{1}{2} Re\{c_T ||v_T(x = x_1, \tau)||_2^2\} + \frac{1}{2} Re\{c_T ||v_T(x = x_2, \tau)||_2^2\}, \quad (35)$$

where c.c. stands for the complex conjugate operator and $\|\cdot\|_2$ for the Euclidean norm. These calculations provide a measure of the energy input and dissipation as the ratio $e = E_{\text{in}}/E_{\text{diss}} = 1$ for all the examples we examined, which implies that the local dampers balance the input energy exactly through the appropriate design.

5. Conclusions

In this paper, we have presented an exact analysis of an Euler-Bernoulli beam connected to ground in its interior by local, linear viscous spring-dampers, pinned and harmonically excited at one end or at both ends and having one of four different boundary conditions (pinned, clamped, free, and linear elastic) at the other end. The local damping introduces non-classical damping to the system so that no classical normal modes exist. This study extends earlier works on spatial wave separation induced by such local damping in non-dispersive media, with the aim to further understand complexity-induced wave separation and vibration localization in dispersive media. Our analysis has shown that by tuning the parameter values of the local damping and stiffness elements, spatial wave separation, as well as vibration response localization in the undamped system, can be obtained. Nearly pure traveling waves, instead of pure traveling waves, are observed due to the evanescent local damping and stiffness elements, spatial wave separation, as well as vibration response localization in dispersive media. Our analysis has shown that by tuning the parameter values of the local damping and stiffness elements, spatial wave separation, as well as vibration response localization in the undamped system, can be obtained. The energies are re-distributed to satisfy the condition for spatial separation of traveling and standing waves in the presence of properly tuned coefficients of the damping. In complex notation, we employ the averaged power flow per cycle \cite{[5]} to calculate the energy transfer from $x = 0$ and $x = 1$, giving

$$E_{\text{in}} = \frac{1}{2} Re\{c.c.[\psi_{xx}(x = 0, \tau)]v_T(x = 0, \tau) + \frac{1}{2} Re\{c.c.[\psi_{xx}(x = 1, \tau)]v_T(x = 1, \tau)\}, \quad (34)$$

while the averaged energy dissipation per cycle by the dampers at $x = x_1$ and $x = x_2$ is expressed by

$$E_{\text{diss}} = \frac{1}{2} Re\{c_T ||v_T(x = x_1, \tau)||_2^2\} + \frac{1}{2} Re\{c_T ||v_T(x = x_2, \tau)||_2^2\}, \quad (35)$$

where c.c. stands for the complex conjugate operator and $\|\cdot\|_2$ for the Euclidean norm. These calculations provide a measure of the energy input and dissipation as the ratio $e = E_{\text{in}}/E_{\text{diss}} = 1$ for all the examples we examined, which implies that the local dampers balance the input energy exactly through the appropriate design.

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It is shown that the mechanism of complexity-induced vibration localization is characterized by the elimination of one of the left- or right-propagating wave components in the beam response, and a linearly-varying spatial phase distribution in the region where traveling waves predominate. For the first four systems considered with one single spring-dashpot pair, combinations of values of the stiffness and damping of the local attachment, denoted by $k_{11}^*$ and $c_{11}^*$, respectively, that give necessary and sufficient conditions for separating the traveling and standing waves are numerically computed and plotted as functions of the excitation frequency $\omega$ and the attachment location $x_0$. For some specified values of $\omega$ and $x_0$, when the local damping vanishes, $c_{11}^* = 0$, the entire system is undamped and global resonance is observed. The exact values of $\omega$ and $x_0$ that correspond to the vanishing of the local damping, $k_{11}^* = 0$, for wave separation were analytically and numerically computed. It is shown that, for either a fixed interior support location or excitation frequency, two sets of values are obtained. These values occur in pairs that are close to each other but never coincide, and they increase in an alternating order. Eliminating points in the damping versus support location and frequency curves that correspond to the so-called dips makes these conditions necessary and sufficient for spatial wave separation. For some other specified values of $\omega$ and $x_0$, where the local stiffness $k_{11}^*$ = 0 but the local damping $c_{11}^* \neq 0$, these conditions require non-classical damping to guarantee the realization of wave separation. As an extension to the previous cases, the last example considered demonstrates that vibration confinement through wave separation is still attainable with multiple translational spring-dampers connected to the beam.

This work presents possibilities for new passive control strategies in dispersive structures via the mechanisms of separation of standing and traveling waves (one-way energy propagation) and vibration localization. It lays the foundation for effective designs of passive energy dissipation and vibration energy harvesting strategies.

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Appendix. Constituent functions for the Euler-Bernoulli beam with clamped or free boundary conditions at the right end

The boundary value problem for the Euler-Bernoulli beam with a clamped right boundary can be obtained by modifying the pinned-pinned case in Section 2. Solving the resulting boundary value problem yields the complex coefficients $D_{i,n}$ ($i = 1, 2, n = 1, \ldots, 4$). Following the solution procedure outlined in Section 2 to establish the necessary and sufficient conditions
for vibration localization, the explicit expressions for \( k_{t1}^* \) and \( c_{t1}^* \), analogous to those defined in Eq. (14) – (16), are shown in Eq. (A.1) – (A.3),

\[
k_{t1}^* = -\frac{8e^{2\gamma}x_0^3}{\kappa} \{ \cosh[\gamma(1-2x_0)]\sin(\gamma) + \cosh(\gamma x_0)\sin[\gamma(2-x_0)] \}
- \cosh[\gamma(2-x_0)]\sin(\gamma x_0) + \cosh(\gamma)\sin[\gamma(1-2x_0)] + \sin[2\gamma(1-x_0)]
+ 2 \cosh[\gamma(1+x_0)]\sin[\gamma(1-x_0)] - 2 \cos(\gamma)\sinh(\gamma) - \cos[\gamma(1-2x_0)]\sinh(\gamma)
- \sinh[2\gamma(2-x_0)] - \cos[\gamma(2-x_0)]\sinh[\gamma(1-x_0)]
- 4 \cosh[\gamma(1-x_0)]\sinh[\gamma(1-x_0)] + 5 \cos(\gamma x_0)\sinh(\gamma x_0)
+ 2 \cos[\gamma(1-x_0)]\sinh[\gamma(1+x_0)] + \cos(\gamma)\sinh[\gamma(1-2x_0)] + \sinh[2\gamma(1-x_0)] \},
\]

(A.1)

\[
c_{t1}^* = \frac{16e^{2\gamma}}{\kappa} \{ \cosh(\gamma)\sin[\gamma(1-x_0)] - \cos[\gamma(1-x_0)]\sinh(\gamma) + \sinh(\gamma x_0) \}
- \cosh[\gamma(1-x_0)]\sin[\gamma(1-x_0)] - \cos(\gamma)\sin[\gamma(1-x_0)]
- \cos[\gamma(1-x_0)]\sinh[\gamma(1-x_0)] - \cos(\gamma)\sinh(\gamma x_0) - \cos(\gamma)\sin[\gamma(1-x_0)] \},
\]

(A.2)

where

\[
\kappa = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} = (1 - e^{2\gamma} - e^{2\gamma x_0} - e^{\gamma(2-x_0)}) \cos(\gamma x_0)
+ e^{\gamma x_0} \cos[\gamma(2-x_0)] + \cos[2\gamma(1-x_0)]
- e^{\gamma x_0} \sin(\gamma x_0) + e^{\gamma(2-x_0)} \sin[\gamma(2-x_0)] + e^{2\gamma} \sin[2\gamma(1-x_0)]
+ 4e^{\gamma} \sinh(\gamma x_0) \{ \cos[\gamma(1-x_0)] - \sin[\gamma(1-x_0)] \}^2
+ \left( 1 - e^{2\gamma} + e^{2\gamma - x_0} + e^{\gamma x_0} \cos(\gamma x_0) \right)
- e^{\gamma(2-x_0)} \cos[\gamma(2-x_0)] - e^{2\gamma} \cos[2\gamma(1-x_0)]
- e^{\gamma(2-x_0)} \sin(\gamma x_0) + e^{\gamma x_0} \sin[\gamma(2-x_0)] + \sin[2\gamma(1-x_0)]
+ 4e^{\gamma} \sinh(\gamma x_0) \{ \cos[\gamma(1-x_0)] + \sin[\gamma(1-x_0)] \}^2 \}.
\]

(A.3)

Given the wave separation conditions, the steady-state solution for this case, comparable to Eq. (20), is given by Eq. (A.4)

\[
v_1(x, \tau) = \left[ \frac{\mathcal{A}}{2} e^{-j\gamma x} + \frac{\mathcal{A}}{2} e^{-x} + 2D_{13}^* \sinh(\gamma x) \right] e^{i\sigma \tau}, \quad 0 \leq x \leq x_0^+,
\]

(A.4)

\[
v_2(x, \tau) = \left[ D_{21}^* e^{-j(1-x)} + D_{22}^* e^{j(1-x)} + D_{23}^* e^{-j(1-x)} + D_{24}^* e^{j(1-x)} \right] e^{i\sigma \tau}, \quad x_0^+ \leq x \leq 1,
\]

where

\[
D_{13}^* = \frac{\mathcal{A}}{\phi} \left\{ (1 - j)e^{-jx_0} - (1 - j)e^{-x_0} - \left[ e^{(1-x_0)} - je^{-\gamma(1-x_0)} \right] e^{-j\gamma} + e^{-\gamma} \left[ e^{-j(1-x_0)} - je^{j\gamma(1-x_0)} \right] \right\},
\]

\[
D_{21}^* = \frac{\mathcal{A}}{\phi} \left\{ \left( -j + e^{-\gamma} \right)e^{-jx_0} - (1 - j) \left( e^{\gamma x_0} - e^{-x_0} \right) e^{-j\gamma} + (1 - j) e^{-j\gamma(1-x_0)} \right\},
\]

\[
D_{22}^* = \frac{\mathcal{A}}{\phi} \left[ je^{-j\gamma x_0} + e^{-\gamma} e^{-jx_0} + (1 - j) e^{j\gamma(1-x_0)} \right],
\]

\[
D_{23}^* = \frac{\mathcal{A}}{\phi} \left\{ e^{j(1-x_0)} - je^{-j(1-x_0)} + \left( e^{\gamma x_0} - e^{-x_0} \right) e^{-j\gamma} + (1 - j) e^{-\gamma} e^{-jx_0} \right\},
\]

\[
D_{24}^* = \frac{\mathcal{A}}{\phi} \left\{ je^{j(1-x_0)} - e^{-j(1-x_0)} - j(e^{\gamma x_0} - e^{-x_0}) e^{-j\gamma} - (1 - j) e^{j\gamma x_0} \right\},
\]

(A.5)

and

\[
\phi = 4(1 - j) \{ \cosh(\gamma)\sin[\gamma(1-x_0)] - \cos[\gamma(1-x_0)]\sinh(\gamma) + \sinh(\gamma x_0) \}.
\]

(A.6)

For the case with a free boundary at the right end, upon solving the boundary value problem the necessary and sufficient conditions for vibration localization are obtained as
\[ k_{11}^* = \frac{8e^{2\gamma} \gamma^3}{\kappa} \left\{ -\cosh[\gamma(1 - 2x_0)]\sin(\gamma) + \cosh(\gamma x_0)\sin[\gamma(2 - x_0)] \\
- \cosh[\gamma(2 - x_0)]\sin(\gamma x_0) - \cosh(\gamma)\sin[\gamma(1 - 2x_0)] + \sin[2\gamma(1 - x_0)] \\
- 2 \cos[\gamma(1 + x_0)]\sin[\gamma(1 - x_0)] + 2 \cos(\gamma)\sinh(\gamma) + \cos[\gamma(1 - 2x_0)]\sinh(\gamma) \\
- \sinh(2\gamma) - \cos[2\gamma(1 - x_0)]\sinh(2\gamma) - \cos[\gamma(2 - x_0)]\sinh[\gamma(2 - x_0)] \\
+ 4 \cos[\gamma x_0]\sinh[\gamma(1 - x_0)] + 5 \cos(\gamma x_0)\sinh(\gamma x_0) \\
- 2 \cos[\gamma(1 - x_0)]\sinh[\gamma(1 + x_0)] - \cos(\gamma)\sinh[\gamma(1 - 2x_0)] + \sinh[2\gamma(1 - x_0)] \right\}. \]

\[ c_{11}^* = \frac{16e^{2\gamma} \gamma}{\kappa} \left\{ \cosh(\gamma)\sin[\gamma(1 - x_0)] - \cos[\gamma(1 - x_0)]\sin(\gamma) - \sinh(\gamma x_0) \\
\times \{ \cosh[\gamma(1 - x_0)]\sin(\gamma) + \sin(\gamma x_0) + \cosh(\gamma)\sin[\gamma(1 - x_0)] \\
- \cos[\gamma(1 - x_0)]\sinh(\gamma) - \cos(\gamma)\sinh[\gamma(1 - x_0)] \} \right\}, \]

where
\[ \kappa = \delta_1^2 + \delta_2^2 = \left( 1 - e^{2\gamma} - e^{2\gamma x_0} - e^{\gamma(2 - x_0)} \cos(\gamma x_0) \right) \\
+ e^{\gamma x_0} \cos[\gamma(1 - x_0)] + e^{\gamma(2 - x_0)} \sin[\gamma(2 - x_0)] + e^{2\gamma} \sin[2\gamma(1 - x_0)] \\
- 4e^\gamma \sinh(\gamma x_0) \cos[\gamma(1 - x_0)] - \sin[\gamma(1 - x_0)] \right\}^2 \\
+ \left( 1 - e^{2\gamma} + e^{2\gamma(1 - x_0)} + e^{\gamma x_0} \cos(\gamma x_0) \right) \\
- e^{\gamma(2 - x_0)} \cos[\gamma(2 - x_0)] - e^{2\gamma} \sin[2\gamma(1 - x_0)] \\
- e^{\gamma(2 - x_0)} \sin[\gamma x_0] + e^{\gamma x_0} \sin[\gamma(2 - x_0)] + \sin[2\gamma(1 - x_0)] \\
- 4e^\gamma \sinh(\gamma x_0) \cos[\gamma(1 - x_0)] + \sin[\gamma(1 - x_0)] \right\}^2. \]

Substituting Eqs. (A.7) – (A.9) into the steady-state solution form Eq. (4) and simplifying the resulting equations yield the steady-state solution for this case,
\[ v_1(x, \tau) = \left[ \frac{\overline{A} e^{-j\psi x} + \overline{A} e^{2\overline{A} - \gamma x} + 2D_{13}^* \sinh(\gamma x)}{2} \right] e^{i\omega_0 \tau}, \quad 0 \leq x \leq x^*_0, \quad (A.10) \]
\[ v_2(x, \tau) = \left[ \overline{D}_{21} e^{-j\rho(1 - x)} + \overline{D}_{22} e^{j\rho(1 - x)} + \overline{D}_{23} e^{-j\gamma(1 - x)} + \overline{D}_{24} e^{j\gamma(1 - x)} \right] e^{i\omega_0 \tau}, \quad x^*_0 \leq x \leq 1, \]

where
\[ D_{13}^* = \frac{\overline{A}}{\psi} \left\{ - (1 - j)e^{-j\psi x_0} + (1 - j)e^{-\gamma x_0} - \left[ e^{\gamma(1 - x_0)} - je^{-\gamma(1 - x_0)} \right] e^{-j\rho x} + e^{-\gamma [e^{-j\rho(1 - x_0)} - je^{-j\rho(1 - x_0)}]} \right\}, \]
\[ D_{21}^* = \frac{\overline{A}}{\psi} \left\{ (e^\rho + je^{-\gamma}) e^{-j\psi x_0} + (1 - j) \left( e^{\gamma x_0} - e^{-\gamma x_0} \right) e^{-j\rho x} - (1 + j) e^{-j\rho (1 - x_0)} \right\}, \]
\[ D_{22}^* = \frac{\overline{A}}{\psi} \left\{ je^{-j\psi x_0} + e^{-\gamma} e^{-j\psi x_0} - (1 + j) e^{j\rho (1 - x_0)} \right\}, \]
\[ D_{23}^* = \frac{\overline{A}}{\psi} \left\{ e^{j\rho (1 - x_0)} - je^{-j\rho (1 - x_0)} + \left( e^{\gamma x_0} - e^{-\gamma x_0} \right) e^{-j\rho x} - (1 - j) e^{-\gamma e^{-j\rho x_0}} \right\}, \]
\[ D_{24}^* = \frac{\overline{A}}{\psi} \left\{ je^{j\rho (1 - x_0)} - e^{j\rho (1 - x_0)} - j(e^{\gamma x_0} - e^{-\gamma x_0}) e^{-j\rho x} + (1 - j) e^{j\rho e^{-j\rho x_0}} \right\}, \]

and
\[ \psi = 4(1 - j) \{ \cosh(\gamma)\sin[\gamma(1 - x_0)] - \cos[\gamma(1 - x_0)]\sin(\gamma) - \sinh(\gamma x_0) \}. \]

In the last two examples we considered an Euler-Bernoulli beam with a compliant boundary at its right end and two viscoelastic supports in its interior, respectively. However, the necessary and sufficient conditions on vibration localization and the final steady-state solution are too lengthy to append. Therefore, we resort to numerical approaches to produce the results given in Sections 4.3 and 4.4.